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Abstract: *We show that the pricing model based on geometric fractional Brownian motion with Hurst index $H > \frac{1}{2}$ behaves to certain extend as a process with bounded variation. If one should hedge an European contingent claim in this pricing model, then the hedging strategy is as if the pricing model was driven by a process with bounded variation. This allows us to construct new arbitrage strategies in this model. On the other hand our findings may be useful in connection to the corresponding pricing model with transaction costs.*

AMS subject classifications: 91B28, 91B70, 60G15, 60H05

Keywords: Arbitrage, pricing by hedging, geometric fractional Brownian motion, stochastic integrals

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1 Introduction

1.1 Motivation

We show how European options with convex payoffs can be hedged in geometric fractional Brownian motion market model. We assume that one can use continuous trading for hedging, the interest rate is equal to zero, and there are no transaction costs.

We shall work with the following market model: the *bond* B is constant $B_t = 1$ for all $t \in [0,T]$, and *stock* S is a geometric fractional Brownian motion:

$$S_t = S_0 e^{B_t^H}$$

with fractional Brownian motion B^H, $H > \frac{1}{2}$: here B^H is a continuous centered Gaussian process with covariance

$$E[B_s^H B_t^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \ s,t \in [0,T] \text{ and } H \in (0,1).$$

The parameter H allows to include the standard Brownian motion W to the fBm family: the process $B^{\frac{1}{2}}$ is a standard Brownian motion.

The standard Brownian motion is a martingale, but it is well-known that when the parameter $H \neq \frac{1}{2}$, then fBm process B^H is not even a semimartingale (see [11]). Since fractional Brownian motion is not a semimartingale, one cannot use the classical theory of stochastic integral to model continuous trading.

If one models continuous trading in the geometric fractional Brownian motion market model by using Riemann - Stieltjes integrals, then one can construct following type of arbitrage strategies: initial capital is equal to zero, and the final value of the portfolio is a non-negative random variable V. One such explicit arbitrage strategy is given by [17, p. 659]. It is not clear, however, if this kind of arbitrage is good enough for hedging options.

On the other hand, if one goes to more realistic market models, and for example includes transaction costs in the market models, then the ideal continuous time trading strategies turn out to be of bounded variation. In this case one can show that geometric fBm models can be economically meaningful in the sense that they do not allow arbitrage possibilities any more (see Guasoni [7], Guasoni et.al [8] for more details). It is also well known that in the case where one can not use continuous time trading, the pricing models with geometric fBm are to some extent arbitrage free (see [3, 1]).

Our motivation comes from the recent works by Bender et al. [1] and Valkeila [21]. In the first work the authors consider a class of models, where the randomness of the risky asset comes from mixed Brownian - fractional Brownian motion. Take this process to be $\epsilon W + B^H$, where W is a standard Brownian motion, B^H is a fBm with index $H \in (\frac{1}{2}, 1)$, and independent of W. If we take the model of the risky asset S^ϵ to be

$$S_t^\epsilon = \exp\{\epsilon W_t + B_t^H - \frac{1}{2}\epsilon^2 t\},$$

then there is a unique *hedging price* for the standard European type of options, provided that one uses so-called *allowed* (in the terminology of [1]) strategies only. If one lets in this model $\epsilon \to 0$, then the limiting price for an European call with strike K is $(S_0 - K)^+$. In the work [21] it is shown that one gets the same limiting price, if one approximates geometric fractional Brownian motion with a sequence of pricing models which are both complete and arbitrage free.

On the other hand, from the hedging point of view the price $(S_0 - K)^+$ for an European call indicates that the hedging strategy should be

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T 1_{\{S_u \geq K\}} dS_u. \tag{1.1}$$

The trading strategy in (1.1) is called *stop-loss-start-gain* in the financial literature. One of the results of this paper is that this strategy with geometric fractional Brownian motion is self-financing, and the integral in (1.1) is an almost sure limit of Riemann sums.

We will show that this is true in the next two sections, where we also explain how the integral is defined. We end the paper with a conclusion. We start with some auxiliary material used to define the stochastic integrals. But first we describe our aims in a more precise way.

1.2 The problem

Throughout the paper $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\mathbb{E}$ stands for the mathematical expectation with respect to probability measure $\mathbb{P}$. Assume $B^H = (B_t^H)_{t \in [0,T]}$ be a standard fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Let $S_t = \exp\{B_t^H\}$ be geometric fractional Brownian motion. Since $F(x) = \exp\{x\} \in C^2(\mathbb{R})$, then by Itô formula([11, Lemma 2.7.1]) we have

$$S_t = 1 + \int_0^t S_u dB_u^H \qquad \text{or} \qquad dS_t = S_t dB_t^H, \tag{1.2}$$

where the stochastic integral can be understood in the sense of Riemann-Stieltjes integral almost surely since the trajectories of the process S_t are Hölder continuous of any order $\lambda < H$ with probability 1 (we refer to [22] for more details).

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. It is well-known that the left derivative of f, f'_- exist a.e. Next we state our main problem.

(i) Does the stochastic integral

$$\int_0^T f'_-(S_t) S_t dB_t^H$$

exist? More precisely, in which sense the integral exists?

(ii) Is it true that for convex function f we have the following Itô formula:

$$f(S_T) = f(S_0) + \int_0^T f'_-(S_t)dS_t? \tag{1.3}$$

It turns out that the integral exists as a generalized Lebesgue-Stieltjes integral, and the Itô formula (1.3) holds. Moreover, the stochastic integral $\int_0^T f'_-(S_t)dS_t$ is the limit of Riemann sums of the form

$$\sum_{k=1}^{n} f'_-(S_{t_{k-1}})(S_{t_k} - S_{t_{k-1}});$$

here $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, and we take the limit as $\max(t_k - t_{k-1}) \to 0$.

The proof of these facts is the topic of the next three sections.

2 Auxiliary results

2.1 Facts on Convex Functions

We recall some results on convex functions. First, recall that every convex function $f : \mathbb{R} \to \mathbb{R}$ has a left-derivative f'_- and a right-derivative f'_+:

$$f'_-(x) = \lim_{h\to 0-} \frac{f(x+h) - f(x)}{h} \text{ and } f'_+(x) = \lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}.$$

The next Theorem gives information about the left-derivative f'_- and right-derivative f'_+.

Theorem 2.1 *[15] The functions f'_- and f'_+ are increasing, respectively left and right-continuous and the set $\{x : f'_-(x) \neq f'_+(x)\}$ is at most countable.*

Moreover, the second derivative of a convex function f exists as a distribution, and first derivative can be represented in terms of the second derivative.

Theorem 2.2 *[15] The second derivative f'' of convex function f exists in the sense of distributions, and it is a positive Radon measure; conversely, for any Radon measure μ on $\mathbb{R}$, there is a convex function f such that $f'' = \mu$ and for any interval I and $x \in \text{int}(I)$ we have the equality*

$$f'_-(x) = \frac{1}{2} \int_I \text{sgn}(x-a)\mu(da) + \alpha_I, \tag{2.1}$$

where α_I is constant and $\text{sgn}(x) = 1$ if $x > 0$ and -1 if $x \leq 0$.

Remark 2.2.1 *If the supp(μ) is compact, then one can globally state that*

$$f'_-(x) = \frac{1}{2} \int \text{sgn}(x-a)\mu(da) \tag{2.2}$$

up to a constant term.

2.2 Pathwise stochastic integration in fractional Besov-type spaces

Fractional Brownian motion is not a semimartingale, and hence the stochastic integral with respect to fractional Brownian motion B^H is not always defined. We shall work with generalized Lebesgue-Stieltjes integrals, and we shall give some details of this construction in this section. For more information see [11, Section 2.1.2].

It turns out that so-called fractional Besov spaces are useful here. We start with some definitions.

Definition 2.1 *Fix* $0 < \beta < 1$.
(i) Let $W_1^\beta = W_1^\beta[0,T]$ *be the space of real-valued measurable functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{1,\beta} := \sup_{0 \le s < t \le T} \left(\frac{|f(t) - f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} du \right) < \infty.$$

(ii) Let $W_2^\beta = W_2^\beta[0,T]$ *be the space of real-valued measurable functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{2,\beta} := \int_0^T \frac{|f(s)|}{s^\beta} ds + \int_0^T \int_0^s \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} duds < \infty.$$

Remark 2.2.2 *The Besov-spaces are closely related to the spaces of Hölder continuous functions. More precisely, for any* $0 < \epsilon < \beta \wedge (1-\beta)$,

$$C^{\beta+\epsilon}[0,T] \subset W_1^\beta[0,T] \subset C^{\beta-\epsilon}[0,T] \quad \text{and} \quad C^{\beta+\epsilon}[0,T] \subset W_2^\beta[0,T].$$

where $C^\gamma[0,T]$ *stands for Hölder continuous functions of order* γ.

Recall that the trajectories of B^H for a.s. $\omega \in \Omega$, any $T > 0$ and any $0 < \gamma < H$ belong to $C^\gamma[0,T]$. This follows from the Kolmogorov continuity theorem. By Remark 2.2.2 we obtain that the trajectories of B^H for a.s. $\omega \in \Omega$, any $T > 0$ and any $0 < \beta < H$ belong to $W_1^\beta[0,T]$.

Denote by $\Gamma(\beta)$ the Gamma-function. Recall the left-sided Riemann-Liouville fractional integral operator I_+^β of order $\beta > 0$:

$$I_{0+}^\beta(f)(s) = \frac{1}{\Gamma(\beta)} \int_0^s f(u)(s-u)^{\beta-1} du.$$

The corresponding right-sided fractional integral operator I_-^β is defined by

$$I_{t-}^\beta(f)(s) = \frac{1}{\Gamma(\beta)} \int_s^t f(u)(u-s)^{\beta-1} du.$$

Remark 2.2.3 *If* $f \in W_1^\beta[0,T]$, *then its restriction to* $[0,t] \subseteq [0,T]$ *belongs to* $I_-^\beta(L_\infty[0,t])$. *Also, if* $f \in W_2^\beta[0,T]$, *then its restriction to* $[0,t] \subseteq [0,T]$ *belongs to* $I_+^\beta(L_1[0,t])$, *where* $I_-^\beta(L_\infty[0,t])$ *(resp.* $I_+^\beta(L_1[0,t])$*) stand for the image of* $L_\infty[0,t]$ *(resp.* $L_1[0,t]$*) by the fractional Riemann-Liouville operator* I_-^β *(resp.* I_+^β*).(For details we refer to [16]).*

Definition 2.2 *Let $f : [0,T] \to \mathbb{R}$ and $0 < \beta < 1$. If $f \in I_+^\beta(L_1[0,T])$(resp. $f \in I_-^\beta(L_\infty[0,T])$ then the Weyl fractional derivatives are defined by*

$$(D_{0+}^\beta f)(x) = \frac{1}{\Gamma(1-\beta)} \left(\frac{f(x)}{x^\beta} + \beta \int_0^x \frac{f(x)-f(y)}{(x-y)^{\beta+1}} dy \right) \boldsymbol{1}_{(0,T)}(x),$$

$$\left(resp. (D_{T-}^\beta f)(x) = \frac{1}{\Gamma(1-\beta)} \left(\frac{f(x)}{(T-x)^\beta} + \beta \int_x^T \frac{f(x)-f(y)}{(y-x)^{\beta+1}} dy \right) \boldsymbol{1}_{(0,T)}(x) \right).$$

For a detailed discussion we refer to [16]. The following proposition clarifies the construction of the stochastic integrals. This approach is by Nualart and Rӑşcanu.

Proposition 2.1 *[13] Let $f \in W_2^\beta[0,T]$, $g \in W_1^{1-\beta}[0,T]$. Then for any $t \in (0,T]$ the Lebesgue integral*

$$\textstyle\int_0^t (D_{0+}^\beta f)(x)(D_{t-}^{1-\beta} g_{t-})(x)dx$$

exists, and we can define the generalized Lebesgue-Stieltjes integral by

$$\int_0^t f dg := \int_0^t (D_{0+}^\beta f)(x)(D_{t-}^{1-\beta} g_{t-})(x)dx.$$

Remark 2.2.4 *The fractional derivative $D_{0+}^\beta f$ also can be denoted by ([11], page 3),*

$$(D_{0+}^\beta f)(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_0^x f(t)(x-t)^{-\beta} dt.$$

So for two functions $f_1, f_2 : [0,T] \to \mathbb{R}$ such that $f_1 = f_2$ a.e. we have

$$\int_0^T f_1 dg = \int_0^T f_2 dg,$$

whenever both side are well-defined.

Remark 2.2.5 *It is shown in [22] when $f \in C^\gamma[0,T]$ and $g \in C^\mu[0,T]$ with $\gamma + \mu > 1$,then the integral $\int_0^T f dg$ exists in the sense of the Proposition 2.1 and coincides with the Riemann-Stieltjes integral.*

The next theorem can be used to study the continuity of the integral.

Theorem 2.3 *[13] Let $f \in W_2^\beta[0,T]$ and $g \in W_1^{1-\beta}[0,T]$. Then we have the estimation*

$$|\int_0^t f dg| \leq \frac{1}{\Gamma(\beta)} \|f\|_{2,\beta} \|g\|_{1,1-\beta}. \tag{2.3}$$

Corollary 2.1 *Let $f, f^n \in W_2^\beta[0,T]$, $\|f^n - f\|_{2,\beta} \to 0$ as $n \to \infty$, and $g \in W_1^{1-\beta}[0,T]$. Then*

$$\int f^n dg \to \int f dg.$$

3 Stochastic integrals and Itô formula

Now we can state the existence result for the stochastic integral. We use the results mentioned in the previous section to show that the integral exists.

Theorem 3.1 *Let $S_t = \exp\{B_t^H\}$ be a geometric fractional Brownian motion, $H \in (\frac{1}{2}, 1)$, $t \in [0,T]$ and $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then the stochastic integral*

$$\int_0^T f_-^{'}(S_t) S_t dB_t^H \tag{3.1}$$

can be understood in the sense of the generalized Lebesgue-Stieltjes integral a.s. $\omega \in \Omega$.

Proof: First assume that $\mathcal{K} := supp(\mu)$ is a compact set. According to Proposition 2.1 the stochastic integral in (3.1) is defined as a generalized Lebesgue-Stieltjes integral if for a selected $\beta \in (1-H, \frac{1}{2})$,

$$\|f_-^{'}(S_t)S_t\|_{2,\beta} < \infty \quad a.s.$$

Obviously,

$$\int_0^T \frac{|f_-^{'}(S_t)||S_t|}{t^\beta} dt \le \max_{t\in[0,T]}(S_t)|f_-^{'}(\max_{t\in[0,T]} S_t)| \int_0^T \frac{1}{t^\beta} dt < \infty \qquad a.s.$$

By the triangular inequality,

$$\int_0^T \int_0^t \frac{|f_-^{'}(S_t)S_t - f_-^{'}(S_s)S_s|}{(t-s)^{\beta+1}} dsdt \le I_1 + I_2,$$

where,

$$\begin{cases} I_1 = \displaystyle\int_0^T \int_0^t \frac{|f_-^{'}(S_t)||S_t - S_s|}{(t-s)^{\beta+1}} dsdt, \\ I_2 = \displaystyle\int_0^T \int_0^t \frac{|S_s||f_-^{'}(S_t) - f_-^{'}(S_s)|}{(t-s)^{\beta+1}} dsdt. \end{cases}$$

Furthermore, using the Hölder continuity property of geometric fractional Brownian motion trajectories one can bound from above I_1 as

$$|I_1| \le |f_-^{'}(\max_{t\in[0,T]} S_t)|C(\omega) \int_0^T \int_0^t \frac{(t-s)^{H-\delta}}{(t-s)^{\beta+1}} dsdt < \infty \qquad a.s.,$$

where $\delta \in (0, H-\beta)$ and C is a almost surely finite random variable such that

$$|S_t - S_s| \le C(\omega)|t-s|^{H-\delta}.$$

We use the representation (2.1) to show I_2 is finite almost surely.

$$|I_2| \le \max_{t\in[0,T]}(S_t) \int_0^T \int_0^t \frac{|f_-^{'}(S_t) - f_-^{'}(S_s)|}{(t-s)^{\beta+1}} dsdt \le I_{2,1} + I_{2,2},$$

where,

$$\begin{cases} I_{2,1} = \max_{t\in[0,T]}(S_t) \int_0^T \int_0^t \int_{\mathcal{K}} \frac{\mathbf{1}_{\{S_s<a<S_t\}}}{(t-s)^{\beta+1}} \mu(da) ds dt, \\ I_{2,2} = \max_{t\in[0,T]}(S_t) \int_0^T \int_0^t \int_{\mathcal{K}} \frac{\mathbf{1}_{\{S_t<a<S_s\}}}{(t-s)^{\beta+1}} \mu(da) ds dt. \end{cases}$$

On the other hand, by *Tonelli's theorem* we have

$$\begin{aligned} \mathbb{E} \int_0^T \int_0^t \int_{\mathcal{K}} \frac{\mathbf{1}_{\{S_s<a<S_t\}}}{(t-s)^{\beta+1}} \mu(da) ds dt &= \int_{\mathcal{K}} \mathbb{E} \left(\int_0^T \int_0^t \frac{\mathbf{1}_{\{S_s<a<S_t\}}}{(t-s)^{\beta+1}} ds dt \right) \mu(da) \\ &\leq \int_{\mathcal{K}} M \mu(da) = M\mu(\mathcal{K}) < \infty, \end{aligned}$$

since μ is a Radon measure and the upper bound M is independent of a (see [12, Lemma A2] and [11], pages 268-269). This implies

$$\int_0^T \int_0^t \int_K \frac{\mathbf{1}_{\{S_s<a<S_t\}}}{(t-s)^{\beta+1}} \mu(da) ds dt < \infty \qquad a.s.$$

Therefore $|I_2| < \infty \quad a.s.$, thus the integral (3.1) exists as a generalized Lebesgue–Stieltjes integral.

Now assume $supp(\mu)$ is not necessarily compact[1]. For any $n \in \mathbb{N}$ define,

$$\mathcal{K}_n = \{\omega \in \Omega | \max_{t\in[0,T]}(S_t) \in [0,n]\},$$

and a new convex function $\tilde{f}_n$ by

$$\tilde{f}_n(x) = \begin{cases} f'_+(0)x + f(0) & \text{if } x < 0, \\ f(x) & \text{if } 0 \leq x \leq n, \\ f'_-(n)(x-n) + f(n) & \text{if } x > n. \end{cases} \tag{3.2}$$

Then $\tilde{f}_n = f$ on the interval $[0,n]$ and moreover $supp(\tilde{\mu}_n) \subset [0,n]$ is compact. Now by the previous argument

$$\int_0^T (\tilde{f}_n)'_-(S_t) S_t dB_t^H$$

is well-defined a.s. on the set $\mathcal{K}_n$. Clearly

$$\int_0^T f'_-(S_t) S_t dB_t^H = \int_0^T (\tilde{f}_n)'_-(S_t) S_t dB^H \qquad a.s. \quad \text{on} \quad \mathcal{K}_n.$$

Since $\Omega = \cup_{n\in\mathbb{N}} \mathcal{K}_n$, this means that $\int_0^T f'_-(S_t) S_t dB_t^H$ is well-defined a.s. $\omega \in \Omega$. □

[1]We thank A-P Perkkiö for the argument how to reduce general case to the compact case.

Remark 3.1.1 *By (1.2) and the Remark 2.2.5 the integral in (3.1) is the same with* $\int_0^T f'_-(S_t)dS_t$.

Remark 3.1.2 *In the same lines one can show the pathwise stochastic integral*

$$\int_0^T f'_+(S_t)S_t dB_t^H$$

is well-defined in the sense of the generalized Lebesgue-Stieltjes integral a.s. Moreover by Theorem 2.1 and Remark 2.2.4,

$$\int_0^T f'_-(S_t)S_t dB_t^H = \int_0^T f'_+(S_t)S_t dB_t^H.$$

Next we consider the Itô formula, which is more interesting for us.

Theorem 3.2 *Let* $S_t = \exp\{B_t^H\}$ *be a geometric fractional Brownian motion with* $H \in (\frac{1}{2}, 1)$, $t \in [0,T]$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be a convex function. Then*

$$f(S_T) = f(S_0) + \int_0^T f'_-(S_t)S_t dB_t^H,$$

where the stochastic integral is understood in the sense of generalized Lebesgue-Stieltjes integral.

Proof: Without loosing generality we can assume supp(μ) corresponds to second derivative of f is compact. If $f \in C^2$ then by Itô formula we have

$$f(S_t) = f(S_0) + \int_0^t f'(S_u)S_u dB_u^H \qquad t \in [0,T], \tag{3.3}$$

where the stochastic integral in the right hand side is limit of Riemann–Stieltjes sums a.s.[11]. We want to show that the equation (3.3) is valid for convex f, where f' is replaced with the left derivative f'_- and the integral is generalized Lebesgue - Stieltjes integral.

Let f be a convex function, and ϕ be a positive C^∞-function with compact support in $(-\infty, 0]$ such that $\int_{-\infty}^0 \phi(y)dy = 1$. Define the functions

$$f_n(x) = n\int_{-\infty}^0 f(x+y)\phi(ny)dy; \qquad n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, $f_n \in C^\infty$is convex, locally bounded. [Note that supp (f_n) is not necessarily compact]. Moreover, f_n converges to f pointwise but f_n' increases to f_-' (see [15, p.221] and [14, p. 210]). In addition, if $g \in C^1$ and has compact support, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x)f_n''(x)dx = \int_{\mathbb{R}} g(x)\mu(dx).$$

Thus by (3.3) we have

$$f_n(S_t) = f_n(S_0) + \int_0^t f_n'(S_u) S_u dB_u^H \qquad t \in [0,T].$$

Obviously $f_n(S_t) \to f(S_t)$ and $f_n(S_0) \to f(S_0)$ a.s. For convergence of stochastic integral by Theorem 2.3, it is sufficient to show

$$\|f_n'(S_t)S_t - f_-'(S_t)S_t\|_{2,\beta} \to 0 \quad n \to \infty \quad a.s.$$

Since

$$\frac{|f_n'(S_t)S_t - f_-'(S_t)S_t|}{t^\beta} \leq \frac{2\max_{t\in[0,T]} S_t |f_-'(\max_{t\in[0,T]} S_t)|}{t^\beta} \in L^1([0,T],dt).$$

The Lebesgue dominated convergence theorem implies

$$\int_0^T \frac{|f_n'(S_t)S_t - f_-'(S_t)S_t|}{t^\beta} dt \to 0 \qquad a.s.$$

Furthermore, by Hölder continuity property of geometric fractional Brownian motion trajectories and mean value theorem one can see

$$\frac{|f_n'(S_t)S_t - f_n'(S_s)S_s|}{(t-s)^{\beta+1}}$$
$$\leq C(\omega)|f_-'(\max_{t\in[0,T]} S_t)|\frac{(t-s)^{H-\delta}}{(t-s)^{\beta+1}} + (\max_{t\in[0,T]} S_t)|f_n''(\theta_\omega)|\frac{(t-s)^{H-\delta}}{(t-s)^{\beta+1}}$$

where θ_ω is between $S_s(\omega)$ and $S_t(\omega)$. Now fix $\omega \in \Omega$, such that the stochastic integral (3.1) is well-defined. Take $\theta_\omega \in [0, \max_{t\in[0,T]} S_t(\omega)]$ be arbitrary, $\epsilon > 0$ and function $\psi_\epsilon \in C^\infty$ with compact support which approximates in uniform norm Dirac delta function δ_{θ_ω}, i.e.

$$\lim_{\epsilon\to 0} \psi_\epsilon(\theta_\omega) = \delta_{\theta_\omega}.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}} \psi_\epsilon f_n''(x)dx &\longrightarrow \int_{\mathbb{R}} \psi_\epsilon \mu(dx) \\ &\longrightarrow \int_{\mathbb{R}} \delta_{\theta_\omega}(x)\mu(dx) = \mu(\theta_\omega) < \infty. \end{aligned}$$

On the other hand, by dominated convergence theorem,

$$\int_{\mathbb{R}} \psi_\epsilon f_n''(x)dx \longrightarrow \int_{\mathbb{R}} \delta_{\theta_\omega}(x) f_n''(x)dx = f_n''(\theta_\omega) \qquad \epsilon \to 0.$$

Therefore $|f_n''(\theta_\omega)| = f_n''(\theta_\omega)$($f_n$'s are convex functions) is uniformly bounded in n and the upper bound is an integrable dominant, so we can deduce again by dominated convergence theorem

$$\|f_n'(S_t)S_t - f_-'(S_t)S_t\|_{2,\beta} \to 0 \quad n \to \infty.$$

□

Remark 3.2.1 *The above results are true also for fractional Brownian motion B^H, when $H > \frac{1}{2}$. If $f : \mathbb{R} \to \mathbb{R}$ is a convex function with left-derivative f'_-, then the integral*

$$\int_0^t f'_-(B_u^H) dB_u^H$$

exists as a generalized Lebesgue-Stieltjes integral. Moreover, we have the change of variables formula

$$f(B_t^H) - f(0) = \int_0^t f'_-(B_u^H) dB_u^H.$$

For example, if $f(x) = |x|$ we have the following version of Tanaka's formula

$$|B_T^H| = \int_0^T sgn(B_u^H) dB_u^H. \tag{3.4}$$

4 Approximation by Riemann-Stieltjes sums

We prove the approximation result for integrals under the assumption that the second derivative of the convex function f has a finite support. As in the Theorem 3.1 one can show that this is not a restriction.

Theorem 4.1 *Assume $T = 1$ and let f be a convex function which positive Borel measure μ corresponding to its second derivative with finite support $\mathcal{K}$, i.e. $\mu(\mathcal{K}) < \infty$. Let $t_i = \frac{i}{n}$; $\quad i = 0, 1, ..., n$. Then,*

$$\sum_{i=0}^n f'_-(S_{t_{i-1}})(S_{t_i} - S_{t_{i-1}}) \xrightarrow{a.s} \int_0^1 f'_-(S_t) dS_t.$$

Proof: Again the key ideas for the proof are representation (2.1) and estimation (2.3). First, note that,

$$\begin{aligned} I_n = \sum_{i=0}^n f'_-(S_{t_{i-1}})(S_{t_i} - S_{t_{i-1}}) - \int_0^1 f'_-(S_t) dS_t \\ = \int_0^1 \left(\sum_{i=0}^n f'_-(S_{t_{i-1}}) 1_{(t_{i-1}, t_i]}(t) - f'_-(S_t) \right) S_t dB_t^H. \end{aligned}$$

Put

$$h_n(t) = \left(\sum_{i=0}^n f'_-(S_{t_{i-1}}) 1_{(t_{i-1}, t_i]}(t) - f'_-(S_t) \right) S_t,$$

then

$$|h_n(t)| \le 2(\max_{t \in [0,1]} S_t) |f'_-(\max_{t \in [0,1]} S_t)| \qquad a.s.$$

Therefore by dominated convergence theorem

$$\int_0^1 \frac{|h_n(t)|}{t^\beta} dt \to 0 \qquad a.s. \qquad \text{as} \quad n \to \infty.$$

Note that $h_n \to 0$ pointwise, since outside a countable set f'_- is a continuous function. Moreover for any $0 \le s \le t \le 1$,

$$
\begin{aligned}
&|S_t \sum_{i=0}^{n} f'_-(S_{t_{i-1}}) 1_{(t_{i-1},t_i]}(t) - S_s \sum_{i=0}^{n} f'_-(S_{t_{i-1}}) 1_{(t_{i-1},t_i]}(s)| \\
&\qquad \le |\sum_{i=0}^{n} f'_-(S_{t_{i-1}}) 1_{(t_{i-1},t_i]}(t)||S_t - S_s| \\
&\qquad + S_s|\sum_{i=0}^{n} f'_-(S_{t_{i-1}}) 1_{(t_{i-1},t_i]}(t) - \sum_{i=0}^{n} f'_-(S_{t_{i-1}}) 1_{(t_{i-1},t_i]}(s)| \\
&\qquad \le |f'_-(\max_{t\in[0,1]} S_t)||S_t - S_s| \\
&\qquad + (\max_{t\in[0,1]} S_t)| \sum_{1\le i\le n, j<i} \left(f'_-(S_{t_{i-1}}) - f'_-(S_{t_{j-1}})\right) 1_{(t_{j-1},t_j]\times(t_{i-1},t_i]}(s,t)|.
\end{aligned}
$$

So, it is enough to find an integrated dominant for the last term w.r.to measure $\frac{1}{(t-s)^{\beta+1}}dsdt$. On the other hand by the representation (2.1) we have

$$
\begin{aligned}
&| \sum_{1\le i\le n, j<i} \left(f'_-(S_{t_{i-1}}) - f'_-(S_{t_{j-1}})\right) 1_{(t_{j-1},t_j]\times(t_{i-1},t_i]}(s,t)| \\
&\quad \le \frac{1}{2} \sum_{1\le i\le n, j<i} \left(\int 1_{\{ S_{t_{j-1}} < a < S_{t_{i-1}}\}} \mu(da)\right) 1_{(t_{j-1},t_j]\times(t_{i-1},t_i]}(s,t) \\
&\quad + \frac{1}{2} \sum_{1\le i\le n, j<i} \left(\int 1_{\{ S_{t_{i-1}} < a < S_{t_{j-1}}\}} \mu(da)\right) 1_{(t_{j-1},t_j]\times(t_{i-1},t_i]}(s,t) \\
&\quad = \frac{1}{2} \int \left(\sum_{1\le i\le n, j<i} 1_{\{ S_{t_{j-1}} < a < S_{t_{i-1}}\}} 1_{(t_{j-1},t_j]\times(t_{i-1},t_i]}(s,t) \right) \mu(da) \\
&\quad + \frac{1}{2} \int \left(\sum_{1\le i\le n, j<i} 1_{\{ S_{t_{i-1}} < a < S_{t_{j-1}}\}} 1_{(t_{j-1},t_j]\times(t_{i-1},t_i]}(s,t) \right) \mu(da) \\
&\qquad \longrightarrow \begin{cases} \frac{1}{2}\int 1_{\{ S_s < a < S_t\}} \mu(da) & \text{on the set } \{S_s < a < S_t\} \ , \\ \frac{1}{2}\int 1_{\{ S_t < a < S_s\}} \mu(da) & \text{on the set } \{S_t < a < S_s\}. \end{cases}
\end{aligned}
$$

which will bring an integrated dominant in both cases (see proof of the Theorem (3.1)). Hence by dominated convergence theorem

$$
\int_0^1 \int_0^t \frac{|h_n(t) - h_n(s)|}{(t-s)^{\beta+1}} dsdt \to 0 \qquad a.s. \qquad \text{as} \quad n \to \infty.
$$

□

5 Conclusions

Shiryaev notes that the properties of no-arbitrage and completeness in a pricing model are not related: there can be completeness both in arbitrage-free models and in models with arbitrage (see [17, p. 661]). One example of a complete market model with arbitrage possibilities is given by Sottinen: he approximates geometric fractional Brownian motion by a complicated 'fractional' tree, which is complete, but can have arbitrage possibilities (see [19] for more details).

Now we can say something more about replication in pricing models with geometric fractional Brownian motion. How much one can replicate in the model with geometric fractional Brownian motion depends on the integral we use. If we use the integration theory of Young, then the integral

$$\int_0^t 1_{\{S_u \geq K\}} dS_u$$

is not defined, since the process $U_u = 1_{\{S_u \geq K\}}$ has infinite p variation for every $p \geq 1$, and it seems that it is difficult to give meaning to the hedging equation

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T 1_{\{S_u \geq K\}} dS_u;$$

but if we interpret the integral as generalized Lebesgue – Stieltjes integral we have shown that this equation has a certain pathwise interpretation as a continuous time hedging strategy. It is also known that if one uses more formal integrals like the Skorohod integral, one can hedge more (see [4] and [10]), but then the economic interpretation of the stochastic integrals as trading strategies becomes more and more difficult (see [2] and [20]).

We can now say something more definitive of the arbitrage/replication issue. If one allows continuous trading, and uses geometric fractional Brownian motion as a model for the risky asset, then the following set of random variables can be hedged with the generalized Lebesgue – Stieltjes integrals:

$$\mathcal{C} = \{f(S_T) : f \geq 0, f \text{ is a linear combination convex functions } \},$$

and the hedging price is given by $f(S_0)$, with the self-financing hedging strategy is $f'_-(S_s)$. Note also that although one can make arbitrage with continuous trading, it is not clear weather this arbitrage is good enough for hedging. On the other hand, the hedging price $f(S_0)$ can be meaningless from the economic point of view.

Sondermann gives an argument why continuous stock prices must have an infinite variation in [18, Remark 6.4], see also the earlier work in [9]. If the driving processes is a standard Brownian motion, then the European call can not be hedged with the stop-loss-start-gain strategy, as it is the case here (cf. (1.1)). If the pricing model is the classical Black – Scholes model, then this stop-loss-start-gain is not self-financing, but instead a term coming from the local time appears. We refer to Sondermann for more details on this. Our results show that out-of-the-money options have zero value, and this is yet another critical point against using pricing models driven by fractional Brownian motion in stochastic finance.

We have shown that there exists models where the price process has infinite variation, and out-of-the-money options have zero value. This is not reasonable from the economic point of view. To exclude this kind of examples in addition to infinite variation one should ask also that non-zero quadratic variation should exist; see [1] for results how one can hedge in some non-semimartingale models, when the quadratic variation exists. Note also that in pricing models with non-zero quadratic variation the arbitrage strategies like the one given by Shiryaev [17] are not anymore arbitrage strategies.

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