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FINITE ELEMENT METHODS FOR PARAMETER DEPENDENT PROBLEMS

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Mika Juntunen: *Finite element methods for parameter dependent problems*; Helsinki University of Technology Institute of Mathematics Research Reports A573 (2009).

Abstract: This thesis develops finite element methods for parameter dependent equations. The interest lies in cases where the nature of the problem and the numerical methods used change with the parameters. The studied examples are the reaction-diffusion problem, the Robin boundary condition, which generalizes the Dirichlet and Neumann conditions, and the Brinkman equation, which generalizes the Stokes and the Darcy equations.

The developed methods depend continuously on the problem parameters and work even for the limiting values. A posteriori estimates are derived for all the proposed methods, taking into account the parameters.

Significant parts of this work are the implementation of the proposed methods and the numerical verfication of the developed theory.

AMS subject classifications: 65N30

Keywords: finite element method, boundary conditions, Nitsche's method, a posteriori estimation, Stokes equation, Darcy equation, Brinkman equation

Mika Juntunen: *Elementtimenetelmiä parametririippuville tehtäville*

Tiivistelmä: Tässä väitöskirjassa on kehitetty menetelmiä parametririippuville tehtäville. Erityisesti tarkastellaan sellaisia tapauksia joissa tehtävän luonne ja käytetyt numeeriset menetelmät selkeästi muuttuvat parametrien mukaan. Tarkasteltavia tehtäviä ovat reaktio-diffuusio ongelma, Robinin reunaehdot, jotka muuttuvat Dirichletin ehdoista Neumanin ehdoiksi, ja Brinkmanin yhtälö, joka käyttäytyy joko Stokesin tai Darcyn tehtävän tapaan.

Työssä kehitetyt menetelmät riippuvat jatkuvasti tehtävän parametreista ja ovat voimassa myös parametrien raja-arvoilla. Lisäksi menetelmille on johdettu a posteriori virhe-estimaattorit joissa riippuvuus tehtävän parametreistä otetaan huomioon.

Merkittävä osa työtä ovat numeeristen menetelmien toteutus ja esitetyn teorian numeerinen testaus.

Avainsanat: elementtimenetelmä, reunaehdot, Nitschen menetelmä, a posteriori arviointi, Stokesin yhtälö, Darcyn yhtälö, Brinkmanin yhtälö

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Vantaa, June 2009 Mika Juntunen

List of included publications

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[A] Mika Juntunen, Rolf Stenberg. On a mixed discontinuous Galerkin method. *Electronic Transactions on Numerical Analysis*, **32** (2008), 17–32.

[B] Mika Juntunen, Rolf Stenberg. Nitsche's method for general boundary conditions. *Mathematics of Computation*, **78**, 267 (2009), 1353–1374. doi:10.1090/S0025-5718-08-02183-2

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[E] Mika Juntunen, Rolf Stenberg. Analysis of finite element methods for the Brinkman problem. *Helsinki University of Technology Institute of Mathematics Research Report* A557 (2009).

[F] Antti Hannukainen, Mika Juntunen, Rolf Stenberg. Computations with finite element methods for the Brinkman problem. *Helsinki University of Technology Institute of Mathematics Research Report* A569 (2009).

Author's contribution

The author of the thesis has written all of the articles, except [C], in collaboration with Prof. Rolf Stenberg. The article [C] is written in collaboration with Prof. Peter Hansbo (Chalmers University of Technology). M.Sc.(Tech.) Antti Hannukainen (TKK, Department of Mathematics and Systems Analysis) is the main developer of the Matlab code used in [F].

1 Introduction

The analysis of the Stokes equations assumes that the viscosity has a positive lower bound. This assumption is perfectly sound for the Stokes flow but in the closely related Brinkman problem, also known as the Darcy-Stokes flow, this assumption is not necessary. In the limit of vanishing viscosity the Brinkman problem simply becomes the Darcy problem. The transition from the Stokes problem to the Darcy problem has various consequences; solution spaces, boundary conditions, and applied elements change, to name a few. The mathematical framework needs to take all these changes into account. One of the main items of this thesis is the finite element analysis of the Brinkman equation. The results are highligted in Section 4.

Another main item of this thesis is enforcing the Robin boundary condition. The classical application of the Robin boundary condition couples the problem parameter and the mesh size. This leads to an ill-conditioned system. In addition, the a rate of convergence is not optimal uniformly with respect to the problem parameter. Section 2 shows how applying the boundary conditions weakly with Nitsche's method improves the situation.

The boundary conditions are also studied in the case of singularly perturbed problems. Here the difficulty is that the essential boundary condition becomes the natural condition in the limit. Section 3 shows how the transition is naturally incorporated into the weak form with Nitsche's method.

Error analysis plays a prominent role in this thesis. The focus is on the residual based a posteriori error estimates that are readily computable once the solution is available. Especially the effect of the problem parameters on the a posteriori indicators is thoroughly studied. Section 5 extracts the a posteriori results derived in the included publications and highlights the main points and similarities.

The finite element method is a numerical method. This is a luxury that needs to be exploited in the analysis. That is to say, the derived theory can, and must, be tested computationally. All the methods and theorems derived in this thesis have passed numerous computational tests and the results are well in line with the theory, see the included publications.

Finally, Section 6 relates the included publications to the course of this thesis and draws conclusions.

2 The Robin boundary condition

Consider the Poisson problem with the Robin boundary condition

$$-\Delta u = f \text{ in } \Omega \subset \mathbb{R}^N, \tag{1}$$

$$\epsilon \partial_n u + u = 0 \text{ on } \partial\Omega, \tag{2}$$

where ∂_n denotes the normal derivative and the parameter $0 \leq \epsilon \leq \infty$. The limiting values of ϵ yield the pure Dirichlet and Neumann boundary conditions respectively:

$$\epsilon = 0 \qquad \Rightarrow \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

$$\epsilon \to \infty \qquad \Rightarrow \qquad \partial_n u = 0 \text{ on } \partial\Omega. \tag{4}$$

Assume a shape regular partitioning $\mathcal{C}_h$ of the domain Ω, where h denotes the maximum size of the elements. The solution space for the problem is

$$V_h = \{v \in H^1(\Omega) \mid v_K \in P_k(K)\ \forall K \in \mathcal{C}_h\}, \tag{5}$$

where P_k denotes the polynomials of order k and K denotes an element in the partitioning. The classical finite element formulation of this problem is: Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v) + \epsilon^{-1} \langle u_h, v \rangle = (f, v) \quad \forall v \in V_h, \tag{6}$$

where $(\cdot,\cdot)$ denotes the L^2-inner product in Ω and $\langle \cdot,\cdot \rangle$ that on the boundary $\partial\Omega$.

In the above formulation the parameter ϵ and the mesh size h are coupled. If $\epsilon \to 0$ the weak form and the discrete system become ill-conditioned [3]. This affects the accuracy of the method. For a smooth solution the a priori error estimate for the problem is [8]

$$\|\nabla(u-u_h)\|_0 + \epsilon^{-1/2}\|u-u_h\|_{0,\partial\Omega} \leq Ch^k(1+\epsilon^{-1/2}h_{\partial\Omega}^{1/2}), \tag{7}$$

where $h_{\partial\Omega}$ denotes the size of the largest element on $\partial\Omega$. Thus, the a priori estimate is not optimal until the mesh length is of the order of ϵ, that is, until $h_{\partial\Omega} \leq C\epsilon$. Above and in what follows the constants C and C_i are generic constants independent of the mesh size and the problem parameters.

To remedy the coupling above the Robin boundary condition is enforced using Nitsche's method [B]. In [15, 12, 17] this is done for the Dirichlet boundary condition. For the Robin boundary the formulation is: Find $u_h \in V_h$ such that

$$\mathcal{B}_h(u_h, v) = (f, v) \quad \forall v \in V_h, \tag{8}$$

where

$$\begin{aligned}\mathcal{B}_h(u_h, v) = (\nabla u_h, \nabla v) + \sum_{E\in\Gamma_h} \Big\{ &- \frac{\gamma h_E}{\epsilon + \gamma h_E}\big(\langle \partial_n u_h, v\rangle_E + \langle \partial_n v, u_h\rangle_E\big) \\ &+ \frac{1}{\epsilon+\gamma h_E}\langle u_h, v\rangle_E - \frac{\epsilon\gamma h_E}{\epsilon+\gamma h_E}\langle \partial_n u_h, \partial_n v\rangle_E \Big\}.\end{aligned} \tag{9}$$

Above Γ_h denotes a partitioning of the boundary $\partial\Omega$ and E denotes a boundary element. For stability assume that the parameter $\gamma > C_I^{-1}$. The constant C_I depends on the inequality

$$h_E\|\partial_n v\|_{\partial K} \leq C_I\|\nabla v\|_K \quad \forall v \in V_h. \tag{10}$$

The above weak form is a generalization of the Dirichlet and Neumann problems and the formulation reduces to these two cases; $\epsilon = 0$ gives Nitsche's method for the Dirichlet boundary conditions

$$\mathcal{B}_h(u_h, v) = (\nabla u_h, \nabla v) + \sum_{E\in\Gamma_h} \Big\{ - \big(\langle \partial_n u_h, v\rangle_E + \langle \partial_n v, u_h\rangle_E \big) + \frac{1}{\gamma h_E} \langle u_h, v\rangle_E \Big\} \tag{11}$$

and $\epsilon \to \infty$ gives the (almost) classical Neumann problem

$$\mathcal{B}_h(u_h, v) = (\nabla u_h, \nabla v) + \sum_{E\in\Gamma_h} \Big\{ - \langle \partial_n u_h, \partial_n v\rangle_E \Big\}. \tag{12}$$

The extra term above does not have effect on the solution. In the Neumann case the data needs to satisfy $f \in L^2_0(\Omega)$.

In addition, $\gamma = 0$ yields the classical formulation (6).

The essential advantage of Nitsche's method over the classical formulation is that it decouples the mesh size and the parameter ϵ. Hence, neither the weak form nor the discrete system become ill-conditioned even if the parameter ϵ reaches the limiting values. The a priori estimate is optimal independent of ϵ. For a smooth solution it holds

$$\|\nabla(u-u_h)\|_0 + \Big(\sum_{E\in\Gamma_h} \frac{1}{\epsilon + h_E} \|u-u_h\|^2_{0,E} \Big)^{1/2} \le Ch^k. \tag{13}$$

3 Singular perturbations

Consider the reaction-diffusion problem

$$-\epsilon^2 \Delta u + u = f \text{ in } \Omega, \tag{14}$$

$$u = 0 \text{ on } \partial\Omega, \tag{15}$$

with the parameter $0 \le \epsilon \le C < \infty$. The solution is sought in the space $\epsilon H^1_0(\Omega) \cap L^2(\Omega)$. Hence, the type of the boundary condition depends on the parameter ϵ. For $\epsilon > 0$ one has the homogenous Dirichlet boundary condition. In the limit $\epsilon = 0$, the solution is the L^2-projection of the load f and the boundary condition disappers.

In classical finite element methods removing the essential boundary conditions is problematic since it requires altering the solution space. Using Nitche's method the boundary conditions are enforced in the bilinear form. Thus, modifying the conditions equals simply modifying the weak form. The proposed method is: Find $u_h \in V_h$ such that

$$(u_h, v) + \epsilon^2 \Big((\nabla u_h, \nabla v) + \sum_{E\in\Gamma_h} \Big[- \langle \partial_n u_h, v\rangle_E - \langle \partial_n v, u_h\rangle_E + \gamma h_E^{-1} \langle u_h, v\rangle_E \Big] \Big) = (f, v) \quad \forall v \in V_h. \tag{16}$$

For stability assume that $\gamma > C_I$, see (10).

The factor $\epsilon^2\gamma h_E^{-1}$ enforces the boundary conditions in the above formulation. As long as $\epsilon \ll h$, the solution of Nitsche's method adapts to the L^2-projection without the boundary conditions. As the ratio ϵ/h grows, the boundary conditions are gradually forced into the system. See Figure 5 in [F] for visualization of this behaviour.

Similar results are observed also for the Brinkman problem [C, E, F] where the boundary condition changes from the essential Stokes condition to the natural Darcy condition as the problem parameter vanishes.

4 The Brinkman problem

Consider the scaled problem

$$-t^2\Delta \boldsymbol{u} + \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \tag{17}$$

$$\nabla \cdot \boldsymbol{u} = g \text{ in } \Omega, \tag{18}$$

where the parameter $0 \le t \le C < \infty$. In the limit $t = 0$, the equations yield the Darcy problem, and for $t \approx 1$ they are the Stokes problem. Hence, the entire nature of the problem changes as the parameter reaches the limiting values.

The problem stated above lacks the boundary conditions. Assume the conditions are homogenous. For the Stokes problem, that is, for $t > 0$ they are

$$\text{essential: } \boldsymbol{u} = \boldsymbol{0}, \tag{19}$$

$$\text{natural: } t^2\partial_n \boldsymbol{u} + p\boldsymbol{n} = \boldsymbol{0}, \tag{20}$$

where $\boldsymbol{n}$ is the outer unit normal and $\partial_n \boldsymbol{u} = \nabla \boldsymbol{u} \cdot \boldsymbol{n}$. For the Darcy problem, that is, in the limit $t = 0$ they are

$$\text{essential: } p = 0, \tag{21}$$

$$\text{natural: } \boldsymbol{u} \cdot \boldsymbol{n} = 0. \tag{22}$$

The essential boundary conditions become the natural ones, and vice versa, as $t \to 0$. In addition, the problem is singularly perturbed.

In the following the boundary conditions on the whole $\partial\Omega$ are

$$\boldsymbol{u} = \boldsymbol{0} \text{ for } t > 0, \tag{23}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ for } t = 0. \tag{24}$$

For compatibility assume that $g \in L_0^2(\Omega)$ and to obtain a unique pressure assume that $p \in L_0^2(\Omega)$.

Both Stokes and Darcy problems are saddle point problems, therefore the norms and solution spaces have to be designed carefully taking into account both limiting problems. The idea in the construction of the solution spaces is to first choose the norm and then define the space such that the norm is finite. For the elliptic velocity part the energy norm is

$$\|\boldsymbol{v}\|_t^2 = t^2\|\nabla \boldsymbol{v}\|_0^2 + \|\boldsymbol{v}\|_0^2. \tag{25}$$

The solution space $\boldsymbol{V}$, taking into account the boundary condition, is the completion of $[C_0^\infty(\Omega)]^N$ with respect to the energy norm. Thus $\boldsymbol{V} = [tH_0^1(\Omega)\cap L^2(\Omega)]^N$. Notice that the boundary condition disappers in the limit $t = 0$ as the boundary condition changes from essential to natural.

The solution space for the pressure is

$$Q = \{q \in L_0^2(\Omega) \mid |||q|||_t < \infty\}, \tag{26}$$

where the norm is

$$|||q|||_t = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{\langle \boldsymbol{v}, \nabla q\rangle_{V\times V^*}}{\|\boldsymbol{v}\|_t}. \tag{27}$$

Above $\langle\cdot,\cdot\rangle_{V\times V^*}$ denotes the duality pairing. For $t = 0$ the norm is $|||q|||_t = \|\nabla q\|_0$, and for $0 < t \leq C$ it holds

$$C_1\|q\|_0 \leq |||q|||_t \leq C_2 t^{-1}\|q\|_0. \tag{28}$$

The practical interpretation of the space is $Q = L_0^2(\Omega)$ for $t = 0$ and $Q = H^1(\Omega)\cap L_0^2(\Omega)$ for $t > 0$.

The above definition of the norms and spaces ensures that Brezzi's conditions [7] hold, which implies that the saddle point problem is stable and has a unique solution.

The requirements for the solution spaces change places as $t \to 0$. For the velocity the space grows from $[H_0^1(\Omega)]^N$ to $[L^2(\Omega)]^N$ and for the pressure the space reduces from $L_0^2(\Omega)$ to $H^1(\Omega)\cap L_0^2(\Omega)$. This carries over to the applicable finite elements. Consider the elements constructed of elementwise P_k polynomials. In the sense of convergence rates, a balanced Stokes method has P_k approximation for the velocity and P_{k-1} for the pressure. In the Darcy limit the balanced approximations are P_k for the pressure and P_{k-1} for the velocity. Therefore equal order approximations seem the best option for the range $0 \leq t \leq C$.

The saddle point nature of the Brinkman problem imposes requirements on the finite elements; they need to satisfy Brezzi's conditions as well. The mathematical framework proposed above is closely related to the Stokes framework. Thus, transferring well-known Stokes elements to Brinkman framework appears to be a good way of designing elements. Using the 'Pitkäranta-Verfürth'-trick [16, 18] and the discrete counterpart of the pressure norm (27)

$$|||q|||_{t,h}^2 = \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2}\|\nabla q\|_{0,K}^2 \tag{29}$$

it is possible to show that the MINI element [2] is stable in the Brinkman problem. Another way of designing stable equal order methods is to stabilize the formulation [6, 14, 11, 10].

5 A posteriori error analysis

The a posteriori estimators studied in this thesis are such that the error in the energy norm is bounded with the sum of the elementwise indicators. The elementwise indicators are readily computable and depend only on the finite element solution and the given data.

It is also important that the estimator is accurate. For the indicators shown below it holds that they are bounded with the sum of the error in the energy norm and the possible data projection error. The constants C_i are indepentent of the mesh size and the problem parameters. In this sense the estimators are sharp.

For the Robin boundary condition (8) the elementwise indicator is

$$\begin{aligned} E_K(u_h)^2 = {} & h_K^2 \|\Delta u_h + f\|_{0,K}^2 + h_K \|[\![\partial_n u_h]\!]\|_{0,\partial K \setminus \partial\Omega}^2 \\ & + \frac{h_K}{(\epsilon + \gamma h_K)^2} \|\epsilon \partial_n u_h + u_h\|_{0,\partial K \cap \partial\Omega}^2. \end{aligned} \tag{30}$$

It is a generalization of the Dirichlet and Neumann estimators; for $\epsilon = 0$ this yields the estimator of Nitsche's method for the Dirichlet conditions

$$E_K(u_h)^2 = h_K^2 \|\Delta u_h + f\|_{0,K}^2 + h_K \|[\![\partial_n u_h]\!]\|_{0,\partial K \setminus \partial\Omega}^2 + h_K^{-1} \|u_h\|_{0,\partial K \cap \partial\Omega}^2 \tag{31}$$

and for $\epsilon \to \infty$ the indicator becomes

$$E_K(u_h)^2 = h_K^2 \|\Delta u_h + f\|_{0,K}^2 + h_K \|[\![\partial_n u_h]\!]\|_{0,\partial K \setminus \partial\Omega}^2 + h_K \|\partial_n u_h\|_{0,\partial K \cap \partial\Omega}^2, \tag{32}$$

which is the usual indicator for the Neumann problem.

For the reaction-diffusion problem (14)-(15), without Nitsche's method, the elementwise indicator is

$$E_K(u_h)^2 = \frac{h_K^2}{\epsilon^2 + h_K^2} \|\epsilon^2 \Delta u_h - u_h + f\|_{0,K}^2 + \frac{h_K}{\epsilon^2 + h_K^2} \|[\![\epsilon^2 \partial_n u_h]\!]\|_{0,\partial K \setminus \partial\Omega}^2 \tag{33}$$

and in the limit $\epsilon = 0$ it reduces to

$$E_K(u_h)^2 = \| - u_h + f\|_{0,K}^2 \tag{34}$$

which is the indicator of the L^2-projection.

For the Brinkman problem (17)-(18) with boundary conditions (23)-(24) the indicator is

$$\begin{aligned} E_K(\boldsymbol{u}_h, p_h)^2 = {} & \frac{h_K^2}{t^2 + h_K^2} \|t^2 \Delta \boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}^2 \\ & + (t^2 + h_K^2) \|\nabla \cdot \boldsymbol{u}_h - g\|_{0,K}^2 \\ & + \frac{h_K}{t^2 + h_K^2} \|[\![t^2 \partial_n \boldsymbol{u}_h]\!]\|_{0,\partial K \setminus \partial\Omega}^2 + \frac{t^2 + h_K^2}{h_K} \|\boldsymbol{u}_h \cdot \boldsymbol{n}\|_{0,\partial K \cap \partial\Omega}^2. \end{aligned} \tag{35}$$

This is a generalization of the Stokes and Darcy indicators. In the Darcy limit $t = 0$ it reduces to the usual Darcy indicator

$$\begin{aligned} E_K(\boldsymbol{u}_h, p_h)^2 = {} & \| - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}^2 + h_K^2 \|\nabla \cdot \boldsymbol{u}_h - g\|_{0,K}^2 \\ & + h_K \|\boldsymbol{u}_h \cdot \boldsymbol{n}\|_{0,\partial K \cap \partial\Omega}^2. \end{aligned} \tag{36}$$

For $t > 0$ the boundary condition is $\boldsymbol{u}_h = 0$ on $\partial\Omega$ thus the last term can be neglected. For $t \approx 1$ the indicator is

$$E_K(\boldsymbol{u}_h, p_h)^2 \approx h_K^2 \|t^2 \Delta \boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}^2 + \|\nabla \cdot \boldsymbol{u}_h - g\|_{0,K}^2 \\ + h_K \|[\![t^2 \partial_n \boldsymbol{u}_h]\!]\|_{0,\partial K \setminus \partial\Omega}^2, \tag{37}$$

which, in turn, is the usual Stokes indicator.

6 Conclusions

[A] The article is a continuation of the authors master's thesis on solving contact problems with Nitsche's method. It studies the mixed discontinuous Poisson problem where the continuity is enforced using Nitsche's method. The analysed method is the stabilized Bassi-Rebay (SBR) method [4, 5]. The given analysis of the SBR method is simpler and more straightforward. Particularly interesting is the application of the Helmholtz decomposition [1, 9, 13] in the proof of the a posteriori estimator. The advantage of the Helmholtz decomposition is that the saturation assumption is not needed. In addition, one can use the well known Clément interpolation, even though the method is discontinous.

[B] The article was developed simultaneously with [A]. It studies enforcing the Robin boundary condition in a weak sense using Nitsche's method. The proposed method generalizes enforcing the Dirichlet and Neumann conditions and this generalization carries over to the error analysis. The main advantage over the classical approach is that in the proposed method the mesh size and the problem parameter are not coupled.

[C] The article was written while the author was visiting Prof. Peter Hansbo at the Chalmers University of Technology. It discusses solving the Brinkman equation with a stabilized low order method. The article focuses on a posteriori analysis and on studying how Nitsche's method enhances the solution close to the boundaries near the Darcy limit.

[D] The article derives an a posteriori estimate for the reaction-diffusion problem. The proposed elementwise indicator depends continuously on the diffusion coefficient. This article is a spin-off of the studies on the Brinkman problem and the results are expanded in [E].

[E] The article continues the studies on the Brinkman problem started in [C]. The article gives a complete finite element analysis of the Brinkman problem; applicable in both the Stokes and Darcy limits. The analysis contains not only a novel mathematical framework but also the a priori and a posteriori error analysis. The studied finite elements are the generalized MINI element [2] and the stabilized equal order methods [6, 14, 11, 10].

[F] The article completes the analysis of [E] with extensive numerical computations. The results suggest that the equal order methods, such as the MINI element and the stabilized methods, give good approximation to the Brinkman problem in both the Stokes and Darcy limits. The article also continues the studies on applying boundary conditions with Nitsche's method in sigularly perturbed problems, which started in [C].

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