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AN L_p-THEORY FOR STOCHASTIC INTEGRAL EQUATIONS

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In the case where A_p is the Laplacian, with $B = \mathbf{R}^n$, sharp regularity results are obtained.

AMS subject classifications: 60H15, 60H20, 45N05

Keywords: stochastic integral equations, stochastic fractional differential equation, regularity, nonnegative operator, Volterra equation, singular kernel

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WOLFGANG DESCH AND STIG-OLOF LONDEN

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and develop an L_p-theory, $2 \leq p < \infty$, for this equation. The solution u is a function of t, ω, x with ω in a probability space and $x \in B$, a σ-finite measure space with positive measure Λ. The kernels $k_1(t)$, $k_2(t)$ are powers of t, i.e., multiples of $t^{\alpha-1}$, $t^{\beta-1}$, with $\alpha \in (0,2)$, $\beta \in (\frac{1}{2}, 2)$, respectively. The mapping A_p is such that $-A_p$ is a nonnegative linear operator of $\mathcal{D}(A_p) \subset L_p(B)$ into $L_p(B)$. The convolution integrals $k_2 \star g^k$ are stochastic Ito-integrals. By combining an approach due to Krylov with transformation techniques and estimates involving fractional powers of $(-A_p)$ we obtain existence and uniqueness results.

In the case where A_p is the Laplacian, with $B = \mathbf{R}^n$, sharp regularity results are obtained.

1. INTRODUCTION

In this paper we analyze the following stochastic parabolic integral equation:

$$u = k_1 * A_p u + \sum_{k=1}^{\infty} k_2 \star g^k + u_0. \tag{1.1}$$

The solution $u = u(t, \omega, x)$ is scalar-valued, $t \geq 0$, $\omega \in \Omega$ (the probability space), $x \in B$, (a σ-finite measure space with positive measure Λ), and

$$\begin{aligned} k_1 * A_p u &= \int_0^t k_1(t-s)(A_p u)(s,\omega,x)\, ds \quad \text{with } k_1(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, \\ k_2 \star g^k &= \int_0^t k_2(t-s) g^k(s,\omega,x)\, dw_s^k \quad \text{with } k_2(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}, \\ u_0 &= u_0(\omega, x). \end{aligned} \tag{1.2}$$

Here, $(w_s^k)_{k=1}^{\infty}$ is a family of independent, scalar-valued Wiener processes, and the integrals $k_2 \star g^k$ are stochastic Ito-integrals. The constants α, β always satisfy (at least) $\alpha \in (0,2)$ and $\beta \in (\frac{1}{2}, 2)$. The parameter β is used to quantify the regularity (or irregularity) of the noise.

As a model for A_p we have in mind the case where B is an open subset of $\mathbf{R}^n$ with smooth boundary, and A_p is a second order elliptic differential operator

$$A_p u = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} b_i(x) D_i u + c(x) u, \tag{1.3}$$

with boundary condition $u(\partial B) = 0$, and with coefficients a_{ij}, b_i, c that are sufficiently smooth. An explicit expression for A_p will, however, be assumed only in

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the examples and when we strive for maximal regularity. In general, A_p will be assumed to be a nonnegative densely defined linear operator of $\mathcal{D}(A_p) \subset L_p(B;\mathbf{R})$ into $L_p(B;\mathbf{R})$.

Our goal is to establish existence and uniqueness of solutions for (1.1) in an L_p-framework with $p \in [2,\infty)$. Regularity results will be stated in terms of fractional powers of A_p (for spatial regularity) and fractional time derivatives as well as Hölder continuity (for time regularity).

Technically we rely on an approach due to Krylov [17], [18], developed for the stochastic partial differential equation

$$\frac{\partial}{\partial t}u(t) = A_p u(t) + \sum_{k=1}^{\infty} g^k(s) dw_s^k,$$

where A_p is an operator of type (1.3). This approach makes use of the Burkholder-Davis-Gundy inequality and sharp estimates for the solution and its spatial gradient. To handle the integral equation (1.1) we combine Krylov's approach with transformation techniques and estimates involving fractional powers of $(-A_p)$. Krylov's approach is very efficient in obtaining maximal regularity, however, it relies on a highly nontrivial Paley-Littlewood inequality [17]. A counterpart of this estimate can be given for general sectorial A_p by straightforward estimates on the Dunford integral, when we allow for an infinitesimal loss of regularity. To obtain maximal regularity — which we do only for the case of the Laplacian in $\mathbf{R}^n$ — a more sophisticated generalization of Krylov's Lemma is required [11].

There is an extensive literature on existence and regularity of solutions of (1.1) in the deterministic case ($k_2 \star g^k dw_s^k$ replaced by a deterministic forcing term $f(t)$, and u_0 independent of ω). We refer in particular to [23], for more regularity results see, e.g., also [7], [8].

Stochastic equations of type (1.1) (with $\beta = 1$) have been considered in a Hilbert space H in [5] (with $\beta = 1$) and [6] (with $\beta \neq 1$), assuming that A_p is self-adjoint, and that the covariance operator Q of the forcing Wiener process commutes with A. This allows the use of spectral resolution. Results on Hölder continuity of the trajectories are obtained. In particular, [6, Theorem 4.2] states sufficient conditions for Hölder regularity in terms of a tradeoff between spatial and time regularity of the stochastic forcing.

An L_2 state-space theory for a Volterra equation perturbed by noise has been developed in [3], extending an approach by [14]. This approach transforms the integral equation (1.1) in an abstract stochastic differential equation in a large state space. Results on existence and regularity of solutions can then be derived from general theorems for analytic semigroups [9].

To our knowledge, [10] is the first attempt to treat the stochastic integral equation (1.1) in an L_p-setting with $p \neq 2$. The present work improves and generalizes the results obtained there. On the other hand, much is known about the stochastically forced heat equation in L_p and even more general Banach spaces, in terms of Krylov's classical setting as well as in terms of the recent advances of stochastic integration theory for Banach space valued functions (e.g., [12], [29]). We will give a short comparison of our regularity results to known results about the stochastic heat equation at the end of this paper. We notice also that in [2] the stochastic heat equation has been generalized by modification of the stochastic forcing. The equation in this work is a parabolic differential equation, but the stochastic forcing is now fractional Brownian motion. This could be remotely compared to our use of the kernel k_2 in (1.1). Like the present paper, [2] is based on Krylov's approach and relies on a suitable adaptation of the Krylov's Paley-Littlewood inequality [17].

2. Outline of paper

In Section 3 we briefly recall the functional analytic tools, in particular some properties of fractional powers of $(-A_p)$ and of D_t. In Section 4 we state our results. Since (1.1) is linear, the contributions of the stochastic forcing and the initial function can be studied separately. Theorem 4.3 and the following remarks are our central result on (1.1) with $u_0 = 0$. Here regularity is stated in terms of fractional powers of $(-A_p)$ and fractional time derivatives. This result requires only that A_p is sectorial, so no maximum regularity can be expected. In Corollary 4.8 we deduce some results on additional regularity in time — in particular on Hölder-continuity — using L_q- and Hölder properties of functions with bounded fractional time derivatives. These results are based on embedding theorems with an epsilon loss of regularity. This epsilon loss of regularity implies, for instance, that the case $\beta = 1$, $p = 2$ is just outside the conditions when we get continuous trajectories. However, in the special case when $B \subset \mathbf{R}^n$ and when $(\lambda I - A_p)^{-1}$ admits a kernel representation, continuity of the trajectories with values in L_2 can be proved in the limiting case $\beta = 1$ (Theorem 4.10).

The contribution of the initial condition, i.e.,

$$u = k_1 * A_p u \ + u_0, \tag{2.1}$$

with u_0 a random variable, is considered in Theorem 4.11. Parts of Theorems 4.3 and 4.11 are combined in Corollary 4.12 to a statement on (1.1). (Obviously, other combinations of the results are possible).

In the case when $B = \mathbf{R}^n$ and when A_p is exactly the Laplacian, Krylov's use of a Paley-Littlewood inequality can be adapted to obtain a maximum regularity result (Theorem 4.14).

Sections 5, 6, 7 and 8 contain the proofs of Theorems 4.3, 4.10, 4.11 and 4.14, respectively. In Section 9 we formulate some examples. In Section 10 we briefly compare the results and the approach given here with known results on the stochastic heat equation, in particular those of [12], [29].

3. Nonnegative Operators, Fractional Powers, and Fractional Integration

In this paper $A_p : \mathcal{D}(A_p) \subset L_p(B;\mathbf{R}) \to L_p(B;\mathbf{R})$ will be a linear operator such that $(-A_p)$ is nonnegative. Regularity in space will be expressed in terms of the fractional powers $(-A_p)^\theta$ of $(-A_p)$, but we give also some relations to interpolation spaces between $L_p(B,\mathbf{R})$ and $\mathcal{D}(A_p)$:

$(X,Y)_{\theta,p}$	real interpolation space of order $\theta \in (0,1)$, $p \in [1,\infty]$,
$(X,Y)_\theta$	real continuous interpolation space of order θ,
$[X,Y]_\theta$	complex interpolation space of order θ.

Regularity in time will be expressed in terms of fractional time derivatives $D_t^\eta f$. In corollaries we will also give regularity results in terms of the following function spaces (containing functions on an interval $[0,T]$ with values in a Banach space X):

$C^\gamma([0,T];X)$	space of Hölder continuous functions with values in X, with Hölder exponent $\gamma \in (0,1)$,
$h^\gamma_{0\to 0}([0,T];X)$	little Hölder-continuous functions with $f(0) = 0$,
$H^\gamma_p([0,T];X)$	Bessel potential space of order γ.

In this section we summarize briefly the definitions and some known results (with adaptations, if necessary) about nonnegative operators, their fractional powers, fractional integration and differentiation, and the interpolation and function spaces mentioned above.

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the space of bounded linear operators on X. Let A be a closed, linear map of $\mathcal{D}(A) \subset X$ into X. The operator $-A$ is said to be nonnegative if $\rho(A)$, the resolvent set of A, contains $(0,\infty)$, and

$$\sup_{\lambda>0} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < \infty.$$

An operator is positive if it is nonnegative and, in addition, $0 \in \rho(A)$. For $\omega \in [0,\pi)$, we define

$$\Sigma_\omega \stackrel{\text{def}}{=} \{\, \lambda \in \mathbf{C} \setminus \{\,0\,\} \mid |\arg\ \lambda| < \omega \,\}.$$

Recall that if $(-A)$ is nonnegative, then there exists a number $\eta \in (0,\pi)$ such that $\rho(A) \supset \Sigma_\eta$, and

$$\sup_{\lambda \in \Sigma_\eta} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{3.1}$$

The spectral angle of $(-A)$ is defined by

$$\phi_{(-A)} \stackrel{\text{def}}{=} \inf\{\, \omega \in (0,\pi] \mid \rho(A) \supset \Sigma_{\pi-\omega},\ \sup_{\lambda\in\Sigma_{\pi-\omega}} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < \infty \,\}.$$

We will rely heavily on the concept of fractional powers of $(-A)$: Let $(-A)$ be a densely defined nonnegative linear operator on X. If $(-A)$ is positive, $(-A)^{-1}$ is a bounded operator, and $(-A)^{-\theta}$ can be defined by integral formulas [4, Ch. 3] or [19, Section 2.2.2]. As usual,

$$(-A)^\theta \stackrel{\text{def}}{=} ((-A)^{-\theta})^{-1}, \quad \theta > 0. \tag{3.2}$$

If $(-A)$ is nonnegative with $0 \in \sigma(-A)$, we proceed as in [4, Ch. 5]: Since $(-A+\epsilon I)$ is a positive operator if $\epsilon > 0$, its fractional power $(-A+\epsilon I)^\theta$ is well defined according to (3.2). We define

$$\mathcal{D}((-A)^\theta) \stackrel{\text{def}}{=} \{\, y \in \bigcap_{0<\epsilon\le\epsilon_0} \mathcal{D}((-A+\epsilon I)^\theta) \mid \lim_{\epsilon\downarrow 0}(-A+\epsilon I)^\theta y \text{ exists } \,\}, \tag{3.3}$$

$$(-A)^\theta y \stackrel{\text{def}}{=} \lim_{\epsilon\downarrow 0}(-A+\epsilon I)^\theta y \quad \text{for } y \in \mathcal{D}((-A)^\theta). \tag{3.4}$$

Lemma 3.1. *Let $-A$ be a nonnegative linear operator on a Banach space X with spectral angle $\phi_{(-A)}$.*

1) *$(-A)^\theta$ is closed and $\overline{\mathcal{D}((-A)^\theta)} = \overline{\mathcal{D}(-A)}$.*
2) *Assume that $\theta\phi_{(-A)} < \pi$. Then $(-A)^\theta$ is nonnegative and has spectral angle $\theta\phi_{(-A)}$.*

Proof. For (1) see [4, p. 109, 142], also [7, Theorem 10]. For (2) see [4, p. 123]. □

Lemma 3.2. *Let $-A$ be a nonnegative linear operator on a Banach space X.*

1) *For $\theta \in (0,1)$,*

$$\big(X, \mathcal{D}(A)\big)_{\theta,1} \subset \mathcal{D}\big((-A)^\theta\big) \subset \big(X, \mathcal{D}(A)\big)_{\theta,\infty},$$

where $\big(X,\mathcal{D}(A)\big)_{\theta,p}$ are the real interpolation spaces between X and $\mathcal{D}(A)$.

2) *If $(-A)^{iy}$ is uniformly bounded for $y \in \mathbf{R}$, $|y| \le 1$, then, for $\theta \in (0,1)$,*

$$\mathcal{D}\big((-A)^\theta\big) = \big[X, \mathcal{D}(A)\big]_\theta, \tag{3.5}$$

the complex interpolation space between $\mathcal{D}(A)$ (with graph norm) and X.

In particular, it follows from (2) that for a large class of elliptic operators $\mathcal{D}\big((-A)^\theta\big)$ is a Sobolev space.

Proof. For (1) and more information on the real interpolation spaces see, e.g., [19, Proposition 2.2.15]. For (2) see [28, p.103]. □

Lemma 3.3. *Let $-A$ be a nonnegative linear operator on a Banach space X with spectral angle $\phi_{(-A)}$. Then for $\eta \in [0, \pi - \phi_{(-A)})$*

$$\sup_{|\arg\ \mu|\le\eta,\, \mu\neq 0} \|(-A)^{\theta}\mu^{1-\theta}(\mu I - A)^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{3.6}$$

Proof. In case $\eta = 0$, see [4, Th. 6.1.1, p. 141]. The general case can be reduced to the case $\mu > 0$ as follows, [13, p. 314]. Write $\mu = \beta e^{i\alpha}$, $\beta > 0$. Then

$$\begin{aligned}
&\sup_{\beta>0}\|(-A)^{\theta}(\beta e^{i\alpha})^{1-\theta}(\beta e^{i\alpha} - A)^{-1}\|_{\mathcal{L}(X)} \\
&= \sup_{\beta>0}\|(-A)^{\theta-1}(\beta e^{i\alpha})^{1-\theta}(-A)(\beta e^{i\alpha} - A)^{-1}\|_{\mathcal{L}(X)} \\
&= \sup_{\beta>0}\|(-A)^{\theta-1}(\beta e^{i\alpha})^{1-\theta}[(-A)(\beta e^{i\alpha} - A)^{-1} \\
&\qquad\qquad + \beta(\beta e^{i\alpha} - A)^{-1}](-A)(\beta - A)^{-1}\|_{\mathcal{L}(X)} \\
&= \sup_{\beta>0}\|(-A)^{\theta-1}[(-A)(\beta e^{i\alpha} - A)^{-1} + \beta(\beta e^{i\alpha} - A)^{-1}]\beta^{1-\theta}(-A)(\beta - A)^{-1}\|_{\mathcal{L}(X)} \\
&= \sup_{\beta>0}\|[(-A)(\beta e^{i\alpha} - A)^{-1} + \beta(\beta e^{i\alpha} - A)^{-1}]\beta^{1-\theta}(-A)^{\theta}(\beta - A)^{-1}\|_{\mathcal{L}(X)} \\
&\le c(\alpha)\sup_{\beta>0}\|\beta^{1-\theta}(-A)^{\theta}(\beta - A)^{-1}\|_{\mathcal{L}(X)},
\end{aligned}$$

where we used the fact that

$$\sup_{\beta>0}\|(-A)(\beta e^{i\alpha} - A)^{-1} + \beta(\beta e^{i\alpha} - A)^{-1}\|_{\mathcal{L}(X)} < \infty,$$

with uniform bound for $|\alpha| \le \eta \in [0, \pi - \phi_{(-A)})$. □

We turn now to fractional differentiation and integration in time:

Definition 3.4. *Let X be a Banach space and $\alpha \in (0, 1)$, let $u \in L_1((0, T); X)$ for some $T > 0$.*

1) *Fractional integration in time is defined by $D_t^{-\alpha}u \stackrel{\text{def}}{=} \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * u$.*
2) *We say that u has a fractional derivative of order $\alpha > 0$ provided $u = D_t^{-\alpha}f$, for some $f \in L_1((0, T); X)$. If this is the case, we write $D_t^{\alpha}u = f$.*

Remark 3.5. Suppose that u has a fractional derivative of order $\alpha \in (0, 1)$. Then $\frac{1}{\Gamma(1-\alpha)}t^{-\alpha} * u$ is differentiable a.e. and absolutely continuous with $D_t^{\alpha}u = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}t^{-\alpha} * u\right)$.

For the equivalence of fractional derivatives in L_p and fractional powers of the realization of the derivative in L_p, we have the following Lemma.

Lemma 3.6. [8, Prop.2] *Let $p \in [1, \infty)$, X a Banach space and define*

$$\mathcal{D}(L) \stackrel{\text{def}}{=} \{u \in W^{1,p}((0, T); X) \mid u(0) = 0\},\ Lu = u' \text{ for } u \in \mathcal{D}(L).$$

Then, with $\beta \in (0, 1)$,

$$L^{\beta}u \;=\; D_t^{\beta}u, \quad u \in \mathcal{D}(L^{\beta}), \tag{3.7}$$

where $\mathcal{D}(L^{\beta})$ coincides with the set of functions u having a fractional derivative in L_p, i.e.,

$$\mathcal{D}(L^{\beta}) = \{\, u \in L_p((0, T); X) \mid \frac{1}{\Gamma(1-\beta)}t^{-\beta} * u \in W_0^{1,p}((0, T); X)\,\}.$$

In particular, D_t^{β} is closed.

We refer to [8] for further properties of the operator D_t^β.

It is convenient to define homogeneous potential spaces over $[0,T]$ as follows (for $\eta \geq 0$, see, e.g., [23, p. 226], [30, p. 28]):

Definition 3.7. *Let X be a UMD-space, let $\eta \in \mathbf{R}$, $p \in (1,\infty)$. For $\eta \geq 0$ we define*

$$\begin{aligned} H_p^\eta(\mathbf{R};X) &\stackrel{\text{def}}{=} \{ f \in L_p(\mathbf{R};X) \mid \text{there exists } g \in L_p(\mathbf{R};X) \text{ such that } \tilde{g} = |\omega|^\eta \tilde{f} \} \\ \|f\|_{H_p^\eta(\mathbf{R};X)} &\stackrel{\text{def}}{=} \|g\|_{L_p(\mathbf{R}:X)}, \end{aligned}$$

where $\tilde{g}$ denotes the Fourier transform of g. For $\eta < 0$ we define

$$\begin{aligned} \tilde{H}_p^\eta(\mathbf{R};X) &\stackrel{\text{def}}{=} \{ f \in L_{1,\mathrm{loc}}(\mathbf{R};X) \mid \text{there exists } g \in L_p(\mathbf{R};X) \text{ such that } \tilde{g} = |\omega|^\eta \tilde{f} \} \\ \|f\|_{H_p^\eta(\mathbf{R}:X)} &\stackrel{\text{def}}{=} \|g\|_{L_p(\mathbf{R}:X)}, \end{aligned}$$

and let $H_p^\eta(\mathbf{R};X)$ be the completion of $\tilde{H}_p^\eta(\mathbf{R};X)$ with respect to this norm. For a bounded interval $[0,T]$, one defines

$$H_p^\eta([0,T];X) \stackrel{\text{def}}{=} \{ h\big|_{[0,T]} \mid h \in H_p^\eta(\mathbf{R};X) \}$$

with

$$\|f\|_{H_p^\eta([0,T];X)} \stackrel{\text{def}}{=} \inf_{h \in S_{0,f}} \|h\|_{H_p^\eta(\mathbf{R}:X)}.$$

Here $S_{0,f} \stackrel{\text{def}}{=} \{ h \in H_p^\eta(\mathbf{R};X) \mid h\big|_{[0,T]} = f \}$.

Note that by this definition any $f \in H_p^\eta([0,T];X)$ is a locally integrable function, even if $\eta < 0$.

Lemma 3.8. *Let X be a UMD-space, and $p \in (1,\infty)$. Let $f \in L_1((0,T);X)$ and $\eta \in (0,1)$. Suppose that $D_t^{-\eta} f \in L_p\big((0,T);X\big)$. Then $f \in H_p^{-\eta}\big((0,T);X\big)$.*

Proof. Define $w(t) = D_t^{-\eta} f(t)$, $0 \leq t \leq T$; $w(t) = 0$, $t \in \mathbf{R}$, $t \notin [0,T]$. Then $w \in L_p(\mathbf{R};X)$. Consider $h(t) \stackrel{\text{def}}{=} \frac{d}{dt}((tI_{\mathbf{R}^+})^{-\eta} * w)$, $t \in \mathbf{R}$ (where I_M denotes the indicator function of a set M). In particular, up to a constant c we have

$$h(t) = \frac{d}{dt}(t^{-\eta} * D_t^{-\eta} f) = cf(t) \quad \text{for } t \in (0,T).$$

Taking Fourier transforms we obtain

$$h(t) = \mathcal{F}^{-1}\{ (is)(is)^{-1+\eta} \tilde{w} \} = \mathcal{F}^{-1}\{ (is)^\eta \tilde{w} \} = \mathcal{F}^{-1}\{ |s|^\eta \frac{(is)^\eta}{|s|^\eta} \tilde{w} \}.$$

By the Marcinkiewicz Multiplier Theorem (e.g., [23, p.215], vector space valued [15, Theorem 1.3]), $m(s) \stackrel{\text{def}}{=} \frac{(is)^\eta}{|s|^\eta}$ is a multiplier on $L_p(\mathbf{R};X)$. So

$$g(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\{ \frac{(is)^\eta}{|s|^\eta} \tilde{w} \} \in L_p(\mathbf{R};X),$$

and $\|g\|_{L_p(\mathbf{R};X)} \leq c\|D_t^{-\eta} f\|_{L_p((0,T);X)}$. Hence $h(t) = \mathcal{F}^{-1}\{ |s|^\eta \tilde{g} \}$ satisfies $h \in H_p^{-\eta}(\mathbf{R};X)$. Thus h restricted to $(0,T)$ is in $H_p^{-\eta}((0,T);X)$ with

$$\|h\|_{H_p^{-\eta}((0,T);X)} \leq c\|D_t^{-\eta} f\|_{L_p((0,T);X)}.$$

∎

4. Results

Throughout this paper, we will make the following assumptions:

Hypothesis 4.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, with $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing right-continuous filtration of σ-algebras satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $\mathcal{P}$ denote the predictable σ-algebra on $\mathbf{R}_+ \times \Omega$ generated by $\{\mathcal{F}_t\}_{t\geq 0}$ and assume that $\{ w_t^k \mid k = 1, 2, ...; t \geq 0 \}$ is a family of independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega, \mathcal{F}, P)$.*

Hypothesis 4.2. *Let B be a σ-finite measure space with positive measure Λ. Fix $p \in [2, \infty)$, and let $-A_p$ be a nonnegative, linear operator of $\mathcal{D}(A_p) \subset L_p(B; \mathbf{R})$ into $L_p(B; \mathbf{R})$. Moreover, $\mathcal{D}(A_p) \cap L_1(B; \mathbf{R}) \cap L_\infty(B; \mathbf{R})$ is dense in $L_p(B; \mathbf{R})$. Let $\alpha \in (0, 2)$ and suppose that*

$$\phi_{-A_p} < \pi\big(1 - \frac{\alpha}{2}\big). \tag{4.1}$$

We will need the extension of A_p to $L_p(B; l_2)$: Denote by l_2 the set of real-valued sequences $g = \{ g^k \}_{k=1}^\infty$ with $|g|_{l_2}^2 \stackrel{\text{def}}{=} \Sigma_{k=1}^\infty |g^k|^2 < \infty$. For a function $g : B \to l_2$, let $\|g\|_p \stackrel{\text{def}}{=} \||g|_{l_2}\|_{L_p(B)}$. We extend A_p to an l_2-valued map by defining

$$\begin{aligned}\mathcal{D}(\tilde{A}_p) = \{ f = \{ f^k \}_{k=1}^\infty \in L_p(B; l_2) \mid \\ f^k \in \mathcal{D}(A_p),\ k = 1, 2, ...; \{ A_p f^k \}_{k=1}^\infty \in L_p(B; l_2) \}\end{aligned}$$

and

$$\tilde{A}_p f = \{ A_p f^k \}_{k=1}^\infty, \quad f \in \mathcal{D}(\tilde{A}_p).$$

By a use of the Khintchine-Kahane inequality (see [20], or [27, p. 115]) it follows that the extension $-\tilde{A}_p$ is a nonnegative map of $\mathcal{D}(\tilde{A}_p)$ into $L_p(B; l_2)$ and that (4.1) holds with ϕ_{-A_p} replaced by $\phi_{-\tilde{A}_p}$. In the sequel we write A_p both for the scalar-valued mapping A_p and for the l_2-valued extension.

Since (1.1) is linear, the contribution of the initial function u_0 and of the stochastic forcing term may be studied separately. The following is our main result concerning the stochastic forcing, with $u_0 = 0$:

Theorem 4.3. *Assume the probability space $(\Omega; \mathcal{F}; P)$ and the Wiener processes $\{ w_t^k \}_{k=1}^\infty$ satisfy Hypothesis 4.1. Let $p \in [2, \infty)$, $A_p : \mathcal{D}(A_p) \subset L_p(B; \mathbf{R}) \to L_p(B; \mathbf{R})$ satisfy Hypothesis 4.2. Let k_1, k_2 be as in (1.2), with $\alpha \in (0, 2)$, $\beta \in (\frac{1}{2}, 2)$. Suppose that for some $T > 0$,*

$$g \in L_p\big((0, T) \times \Omega; \mathcal{P}; L_p(B; l_2)\big). \tag{4.2}$$

a) *Then there exists a unique $u \in L_p\big((0, T) \times \Omega; \mathcal{P}; L_p(B; \mathbf{R})\big)$ such that $k_1 * u \in \mathcal{D}(A_p)$ a.e. on $(0, T) \times \Omega$, and which satisfies*

$$u = A_p(k_1 * u) \ + \ \sum_{k=1}^\infty k_2 \star g^k. \tag{4.3}$$

Here (4.3) is to be understood as an equation in $L_p\big((0, T) \times \Omega; \mathcal{P}; L_p(B; \mathbf{R})\big)$. (Notice also Remark 4.4 below.)

b) *Suppose $\theta \in [0, 1]$ is such that*

$$\beta - \alpha\theta > \frac{1}{2}. \tag{4.4}$$

Then $u \in \mathcal{D}((-A_p)^\theta)$ a.e. on $(0, T) \times \Omega$, and

$$u = -(-A_p)^{1-\theta}(k_1 * (-A_p)^\theta u) \ + \sum_{k=1}^\infty k_2 \star g^k. \tag{4.5}$$

Equality in (4.5) *holds in* $L_p\big((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R})\big)$. *Moreover, the following estimate holds:*

$$\|(-A_p)^\theta u\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R}))} \le c\|g\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B;l_2))}, \tag{4.6}$$

for some constant c, independent of g but depending on A, α ,β, θ, and p.

c) *If $\theta\in[0,1]$ and $\eta\in(-1,1)$ are such that*

$$\beta-\alpha\theta-\eta>\frac{1}{2}, \tag{4.7}$$

then u has a fractional derivative of order η (if $\eta<0$, a fractional integral of order $-\eta$), where fractional differentiation (integration) is to be understood in the space $L_p(\Omega\times B;\mathbf{R})$. Moreover, $D_t^\eta u\in L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathcal{D}(-A)^\theta))$ and satisfies an estimate

$$\|(-A_p)^\theta D_t^\eta u\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R}))} \le c\|g\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B;l_2))}, \tag{4.8}$$

for some constant c, independent of g but depending on A, α ,β, θ, η, and p.

d) *If* (4.7) *holds, and $\eta\notin\{\frac{1}{p},1+\frac{1}{p}\}$, then*

$$(-A_p)^\theta u\in H_p^\eta\big([0,T];\ L_p(\Omega\times B;\mathbf{R})\big). \tag{4.9}$$

e) *With $\eta\in(-1,1)$ such that $\beta-\eta>\frac{1}{2}$, one has*

$$D_t^\eta u=A_p(k_1*D_t^\eta u)\ +D_t^\eta\Big(\sum_{k=1}^\infty k_2\star g^k\Big). \tag{4.10}$$

Before proceeding, we make a few remarks on Theorem 4.3.

Remark 4.4. The infinite series in (4.3) is to be understood by an approximation procedure (c.f., [18]): Suppose $g=\{\,g^k\,\}_{k=1}^\infty\in L_p((0,T)\times\Omega;\mathcal{P};L_p(B;l_2))$ is given. Obviously, an arbitrary g^k is not necessarily bounded. However, by the density statement Lemma 5.1, one may approximate g^k and g by g_j^k, g_j, respectively, where $g_j=\{\,g_j^k\,\}_{k=1}^j$, $g_j^k=0$ for $k>j$, are adapted and such that

$$\|g_j-g\|_{L_p((0,T)\times\Omega:\mathcal{P};L_p(B;l_2))}\to 0,\quad j\to\infty,$$

and such that each g_j^k is bounded in t,ω, and in x. The sums on the right sides of (4.3), (4.5) should be read as

$$\sum_{k=1}^\infty k_2\star g^k \stackrel{\text{def}}{=} \lim_{j\to\infty}\sum_{k=1}^j\int_0^t k_2(t-s)g_j^k(s,\omega,x)\ dw_s^k. \tag{4.11}$$

We show in Lemma 5.3 that this limit exists in $L_p\big((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R})\big)$.

In fact, in the proof of Theorem 4.3 one approximates g by g_j, then obtains the corresponding solution u_j, and finally proves appropriate convergence results. Working with the bounded functions g_j^k avoids technical problems about existence of stochastic integrals.

Remark 4.5. If $-A_p$ admits bounded imaginary powers, (4.5) and Lemma 3.2(2) imply that u takes values in the complex interpolation space $[X,\mathcal{D}(A_p)]_\theta$.

Remark 4.6. In case (c) of Theorem 4.3, if $\eta>0$, one may first apply case (b) and see that $(-A)^\theta u\in L_p((0,T)\times\Omega;L_p(B;\mathbf{R}))$. Then the closedness of $(-A_p)^\theta$ implies that

$$(-A_p)^\theta[D_t^\eta u]=D_t^\eta[(-A_p)^\theta u].$$

Thus, in this case, $(-A_p)^\theta u$ has a fractional derivative of order η.

Remark 4.7. With $\alpha = \beta = 1$ one has by (4.8) for all $\eta > 0$

$$D_t^{-\eta}((-A_p)^{\frac{1}{2}}u) \in L_p\big((0,T); L_p(\Omega \times B; \mathbf{R})\big).$$

Thus, by Lemma 3.8, if $(-A_p)^{\frac{1}{2}}u \in L_1([0,T], L_p(\Omega \times B; \mathbf{R}))$,

$$(-A_p)^{\frac{1}{2}}u \in H_p^{-\eta}\big((0,T); L_p(\Omega \times B; \mathbf{R})\big), \quad \eta > 0. \tag{4.12}$$

By bringing the parameter p into play one may in fact obtain somewhat more than (4.6) or (4.8), in particular statements on Hölder continuity. This we formulate in the following corollary.

Corollary 4.8. *Let the assumptions of Theorem 4.3 hold.*

a) *If $\beta - \alpha\theta - \frac{1}{2} < p^{-1}$, then one has, in addition to* (4.6)*,*

$$(-A_p)^{\theta}u \in L_q\big((0,T); L_p(\Omega \times B; \mathbf{R})\big), \quad \textit{for} \quad p \le q < q_0, \tag{4.13}$$

and, for almost all $\omega \in \Omega$,

$$[(-A_p)^{\theta}u](\cdot,\omega,\cdot) \in L_q\big((0,T); L_p(B; \mathbf{R})\big), \quad \textit{for } p \le q < q_0, \tag{4.14}$$

where

$$q_0 = \frac{p}{1 - p(\beta - \alpha\theta - \frac{1}{2})}.$$

b) *If $p^{-1} \le \beta - \alpha\theta - \frac{1}{2}$, then*

$$(-A_p)^{\theta}u \in L_q\big((0,T); L_p(\Omega \times B; \mathbf{R})\big) \quad \textit{for} \quad q \in [p,\infty), \tag{4.15}$$

and, for almost all $\omega \in \Omega$,

$$[(-A_p)^{\theta}u](\cdot,\omega,\cdot) \in L_q\big((0,T); L_p(B; \mathbf{R})\big), \quad \textit{for } q \in [p,\infty). \tag{4.16}$$

c) *If $p^{-1} < \eta < \beta - \alpha\theta - \frac{1}{2}$, then*

$$(-A_p)^{\theta}u \in h_{0\to 0}^{\eta-\frac{1}{p}}\big([0,T]; L_p(\Omega \times B; \mathbf{R})\big), \tag{4.17}$$

and, for almost all $\omega \in \Omega$,

$$[(-A_p)^{\theta}u](\cdot,\omega,\cdot) \in h_{0\to 0}^{\eta-\frac{1}{p}}\big([0,T]; L_p(B; \mathbf{R})\big). \tag{4.18}$$

Here $h_{0\to 0}^{\gamma}$ are the little-Hölder continuous functions having modulus of continuity γ and vanishing at the origin.

Proof. To prove a), let $p \le q < q_0$ and choose $\eta \in (0, \frac{1}{p})$ such that (4.7) holds and

$$q < \frac{p}{1 - \eta p}.$$

Recall the fact (see [8, p. 420–421]) that if $D_t^{\eta}v \in L_p\big((0,T); X\big)$ for some function $v \in L_1((0,T); X)$, and $\eta p < 1$, then

$$v \in L_q\big((0,T); X\big), \quad 1 \le q < \frac{p}{1 - \eta p}. \tag{4.19}$$

Use this, together with (4.8) and with $X = L_p(\Omega \times B; \mathbf{R})$, to get the first part of a). For the second part observe that (4.8) implies

$$\|D_t^{\eta}\big((-A_p)^{\theta}u\big)\|_{L_p((0,T);L_p(B))} < \infty,$$

for a.a. $\omega \in \Omega$. Combine this with (4.19), taking $X = L_p(B; \mathbf{R})$, to get the second part of a).

To get b), assume that $\beta - \alpha\theta - p^{-1} \ge \frac{1}{2}$ and take any $q \ge 1$. Choose $\eta \in (0, p^{-1})$ sufficiently close to p^{-1}, such that $q < p(1 - \eta p)^{-1}$. Since $\eta < p^{-1}$ we have (4.7). Then apply (4.8) and recall (4.19) to obtain (4.16).

To prove c), observe that ([8, p. 421]) if the fractional derivative f of order η of v satisfies

$$f \in L_p\big((0,T);X\big), \text{with } p^{-1} < \eta, \tag{4.20}$$

then $v \in h_{0\to 0}^{\eta - p^{-1}}([0,T];X)$. □

Remark 4.9. In Corollary 4.8(c) with $\theta = 0$, $\beta = 1$, $p > 2$, one has

$$u \in h_{0\to 0}^{\eta - p^{-1}}\big([0,T];L_p(B;\mathbf{R})\big), \quad \eta \in (\frac{1}{p},\frac{1}{2}). \tag{4.21}$$

Thus, in this case, for a.a. $\omega \in \Omega$, the solution u (and $(-A_p)^\theta u$ for appropriate θ) is Hölder continuous in time, with values in $L_p(B;\mathbf{R})$, (independently of α if $\theta = 0$).

In Remark 4.9, the case $p = 2$ is obviously excluded by (4.21). However, by taking $B \subset \mathbf{R}^n$, and imposing an additional condition on A_p, we have the following result for this case. We give the proof of this result in Section 6.

Theorem 4.10. *Let the assumptions of Theorem 4.3 hold with $p = 2$, $\beta = 1$ and $\theta = 0$. Assume $B \subset \mathbf{R}^n$ with the Lebesgue measure Λ, and suppose that $(\lambda I - A_p)^{-1}$ admits a kernel representation:*

$$\big((\lambda I - A_p)^{-1} f\big)(x) = \int_B \gamma_\lambda(x,y) f(y)\,dy, \quad x \in B,$$

for $f \in L_p(B;\mathbf{R})$, with the kernel γ_λ satisfying a Poisson estimate

$$|\gamma_\lambda(x,y)| \le c|\lambda|^{\frac{n}{m}-1}\,\Psi\left(|x-y|\,|\lambda|^{\frac{1}{m}}\right) \tag{4.22}$$

for λ in a sector $\Sigma_{\pi-\phi}$ such that $\phi + \alpha\pi < \pi$, and some $m > 0$. Here $\Psi : (0,\infty) \to (0,\infty)$ is a continuous nonincreasing function with

$$\int_0^\infty \Psi(r) r^{n-1}\,dr < \infty.$$

Then the solution $u(t,\omega,x)$ of (4.3) satisfies $u \in C\big([0,T];L_2(B;\mathbf{R})\big)$ for a.a. ω with

$$\sup_{0\le t\le T} \|u(t,\cdot,\cdot)\|_{L_2(\Omega\times B)} \le c\|g\|_{L_2((0,T)\times\Omega;L_2(B;l_2))},$$

for some constant c, depending on α.

We refer the reader to [1] and [25], and to the references therein, for treatments of kernel estimates.

We complement Theorem 4.3 with a statement on solutions of (2.1), i.e., the homogeneous integral equation with nonzero initial condition u_0.

Theorem 4.11. *Let α, p, B, A_p and the probability space $(\Omega;\mathcal{P};P)$ be as in Theorem 4.3.*

a) *Suppose*

$$u_0 \in L_p(\Omega;\mathcal{F}_0;L_p(B;\mathbf{R})). \tag{4.23}$$

Then there exists a unique function u_1 such that $u_1(t,\omega,\cdot) \in \mathcal{D}(A_p)$ for $t > 0$, and a.a. $\omega \in \Omega$, and

$$u_1(t) = A_p \int_0^t k_1(t-s)u_1(s)\,ds + u_0. \tag{4.24}$$

b) *For $\theta \in (0,1]$, $t > 0$, and an apriori constant c, independent of θ,*

$$\|(-A_p)^\theta u_1(t,\cdot,\cdot)\|_{L_p(\Omega;L_p(B))} \le ct^{-\alpha\theta}\|u_0\|_{L_p(\Omega;L_p(B))}. \tag{4.25}$$

Thus, u_1 solves (2.1) in the sense that for $t > 0$, and θ such that $\alpha\theta < 1$,

$$u_1(t,\omega,x) = u_0(\omega,x) - (-A_p)^{1-\theta}\int_0^t k_1(t-s)(-A_p)^\theta u_1(s,\omega,x)\, ds. \tag{4.26}$$

Equality in (4.26) holds for $t > 0$ both in $L_p(\Omega; L_p(B;\mathbf{R}))$, and for a.a. $\omega \in \Omega$ in $L_p(B;\mathbf{R})$. In addition,

$$\lim_{t\to 0+} u_1(t) = u_0$$

in $L_p(B;\mathbf{R})$ for a.a. $\omega \in \Omega$ and in $L_p(\Omega; L_p(B;\mathbf{R}))$.

c) *Let $\eta > 0$ be such that $\eta p < 1$. Then*

$$\|D_t^\eta(u_1 - u_0)\|_{L_p((0,T)\times\Omega;L_p(B;\mathbf{R}))} \le c(\eta)\|u_0\|_{L_p(\Omega;L_p(B;\mathbf{R}))}. \tag{4.27}$$

d) *Let $\alpha p > 1$, and let, for some $\hat\mu$ satisfying $1 - \frac{1}{\alpha p} < \hat\mu < 1$,*

$$u_0 \in L_p\Big(\Omega; \big(L_p(B;\mathbf{R}), \mathcal{D}(A_p)\big)_{\hat\mu}\Big). \tag{4.28}$$

Then the solution u_1 of (4.24) satisfies

$$D_t^\alpha(u_1 - u_0) \in L_p\Big((0,T)\times\Omega; L_p(B;\mathbf{R})\Big), \tag{4.29}$$

with the L_p-norm of $D_t^\alpha(u_1-u_0)$ bounded by an apriori constant multiplying the norm of u_0 in the space of (4.28).

In particular, if

$$u_0 \in L_p\Big(\Omega; \mathcal{D}\big((-A_p)^\theta\big)\Big) \ for\ some \ \ \theta > 1 - \frac{1}{\alpha p}, \tag{4.30}$$

then (4.29) holds.

A combination of (4.5), (4.6) of Theorem 4.3 and Theorem 4.11 (a,b) gives the following corollary.

Corollary 4.12. *Let $\alpha \in (0,2)$, $\beta \in (\frac{1}{2},2)$, $\theta \in (0,1]$, $\beta - \alpha\theta > \frac{1}{2}$. Let $p \in [2,\infty)$ and assume that Hypotheses 4.1 and 4.2 are satisfied. Suppose g, u_0 satisfy, respectively, (4.2) and (4.23). Assume $\alpha\theta p < 1$. Then there exists a unique solution u of (1.1) such that*

$$\begin{aligned}
&u \in \mathcal{D}\big((-A_p)^\theta\big), \quad a.e. \ \ on\ (0,T)\times\Omega,\\
&(-A_p)^\theta u \in L_p\big((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R})\big)\\
&u = -(-A_p)^{1-\theta}\big(k_1 * (-A_p)^\theta u\big) + \sum_{k=1}^\infty k_2 \star g^k + u_0,\ t \ge 0,\\
&\|(-A_p)^\theta u\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R}))} \le\\
&\quad c\big[\|g\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B;l_2))} + \|u_0\|_{L_p(\Omega;L_p(B;\mathbf{R}))}\big].
\end{aligned}$$

Remark 4.13. Suppose that, in Corollary 4.12, α, β are large enough, e.g., in case $\theta = 0$, $\beta > \frac{3}{2}$, $\alpha > 1$. Obviously, one may then add a term tv_0 to (1.1), where $v_0 \in L_p(\Omega;\mathcal{F}_0;L_p(B;\mathbf{R}))$ and interpret v_0 as an initial condition $\frac{d}{dt}u(0) = v_0$. We have, for simplicity, taken $v_0 = 0$.

In the special case that A_p is the Laplacian on $L_p(\mathbf{R}^n;\mathbf{R})$, Hypothesis 4.2 is satisfied, and with suitable convolution kernels, Theorem 4.3 can be applied. In addition, we obtain a maximal regularity result in the sense that the strict inequality in (4.4) can be replaced by "$\ge$":

Theorem 4.14. *Let Δ_p denote the Laplacian on $L_p(\mathbf{R}^n;\mathbf{R})$ for $n \geq 1$, $p \in [2,\infty)$. Let $\alpha \in (0,2)$, $\beta \in (\frac{1}{2},2)$, k_1 and k_2 as in (1.2), and let the probability space $(\Omega,\mathcal{F},P)$ and the Wiener processes w_t^k be as in Hypothesis 4.1. Suppose that $\theta \in [0,1)$ is such that*

$$\beta - \alpha\theta \geq \frac{1}{2}. \tag{4.31}$$

Then there exists a constant c depending on n, p, α, β, θ such that for any $g \in L_p((0,T)\times\Omega;\mathcal{P};L_p(\mathbf{R}^n;l_2))$, the solution u of

$$u(t) = \Delta_p(k_1 * u) + \sum_{k=1}^{\infty} k_2 \star g^k \tag{4.32}$$

(according to Theorem 4.3) satisfies the following estimate:

$$\|(-\Delta_p)^\theta u\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(\mathbf{R}^n;\mathbf{R}))} \leq c\|g\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(\mathbf{R}^n;\mathbf{R}))}. \tag{4.33}$$

Remark 4.15. Of course, (4.33) is an analogon to (4.8) in the case of $\eta = 0$ and $\beta - \alpha\theta = \frac{1}{2}$. One is tempted to conjecture that for the Laplacian also (4.10) can be extended to the case that $\beta - \alpha\theta - \eta = \frac{1}{2}$. However, if we take $\theta = 0$ and $\beta - \eta = \frac{1}{2}$, then $D_t^\eta k_2 \star g^k$ is not well defined in $L_p((0,T)\times\Omega\times B;\mathbf{R})$. This can be seen most easily in the case $p = 2$ and $g = g_1 = 1$, where we use that $\beta - \eta = \frac{1}{2}$ to obtain formally

$$D_t^\eta k_2 \star g = c\,\frac{d}{dt}\int_0^t (t-s)^{\frac{1}{2}}\,dw_s.$$

However, for $\epsilon \downarrow 0$, it is easily estimated by Ito's isometry that

$$\int_\Omega \frac{1}{\epsilon^2}\left|\int_0^{t+\epsilon}(t+\epsilon-s)^{\frac{1}{2}}\,dw_s - \int_0^t (t-s)^{\frac{1}{2}}\,dw_s\right|^2 dP(\omega) \to \infty.$$

5. Proof of Theorem 4.3

We begin by proving the density statement referred to earlier.

Lemma 5.1. [18] *Let $p \in [2,\infty)$, $g \in L_p(\mathbf{R}_+\times\Omega;\mathcal{P};L_p(B;l_2))$. Let G be any countable dense subset of $\mathcal{D}(A_p)\cap L_\infty(B;\mathbf{R})\cap L_1(B;\mathbf{R})$.*
Then there exist adapted $\{\,g_j\,\}_{j=1}^\infty$, $g_j \in L_p(\mathbf{R}_+\times\Omega;\mathcal{P};L_p(B;l_2))$, $g_j = \{\,g_j^k\,\}_{k=1}^\infty$, such that $\|g-g_j\|_{L_p(\mathbf{R}_+\times\Omega;\mathcal{P};L_p(B;l_2))} \to 0$, as $j\to\infty$, and such that

$$\begin{aligned} g_j^k &= \sum_{i=1}^{j} I_{\tau_{i-1}^j < t \leq \tau_i^j}(t)\,g_j^{ik}(x), \quad k \leq j,\\ g_j^k &= 0, \qquad k > j, \end{aligned} \tag{5.1}$$

with $g_j^{ik} \in G$ and bounded stopping times $\tau_0^j \leq \tau_1^j \leq .... \leq \tau_j^j$.

Proof. The set of $g \in L_p(\mathbf{R}_+\times\Omega;\mathcal{P};L_p(B;l_2))$ for which the statement holds, is a closed subspace $M \subset L_p(\mathbf{R}_+\times\Omega;\mathcal{P};L_p(B;l_2))$. Then, if $L_p \setminus M \neq \emptyset$, there exists $h \in L_q(\mathbf{R}_+\times\Omega;\mathcal{P};L_q(B;l_2))$; $q^{-1}+p^{-1} = 1$, such that $h \not\equiv 0$, $h(M) = 0$, that is,

$$\int_{\mathbf{R}_+\times\Omega}\int_B (h,g)\;d\Lambda\;dP(\omega)\;dt\;\;= 0, \quad \text{for all } g \in M.$$

Here $(h,g) = \Sigma_{k=1}^\infty h^k g^k$.

Take an arbitrary bounded stopping time $\tau(\omega)$ and fix some k_0. Let $g = \{\,g^k\,\}_{k=1}^\infty$ be defined by

$$g^{k_0} = I_{0<t\leq\tau}\tilde{g}; \quad g^k = 0,\; k \neq k_0,$$

where $\tilde{g} \in G$. Thus $g \in M$. Therefore, $\int_{\mathbf{R}_+\times\Omega} I_{0<t\leq\tau} F(t,\omega)\ dP(\omega)\ dt\ = 0$, where $F(t,\omega) \stackrel{\text{def}}{=} \int_B h^{k_0}(t,\omega,x)\tilde{g}(x)\ d\Lambda$ is a predictable process. Since G is dense and τ, $\tilde{g}$ are arbitrary, it is not difficult to show that then $\int_B h^{k_0}(t,\omega,x)g(x)\ d\Lambda = 0$, a.e. on $\mathbf{R}_+ \times \Omega$, for all $g \in L_p(B;\mathbf{R})$. One concludes that $h^{k_0} = 0$ a.e. on $\mathbf{R}_+ \times \Omega \times B$.

But k_0 was arbitrary and so $h^k = 0$ a.e. for all k. This contradicts the assumption $h \not\equiv 0$, and so Lemma 5.1 follows. □

An essential ingredient to the proof is the following L_p-estimate for stochastic convolutions, obtained by the Burkholder-Davis-Gundy inequality:

Lemma 5.2. *Let $p \in [2,\infty)$, let $\{V(t) \mid t \geq 0\}$ be a family of bounded linear operators $V(t) : \mathcal{D}(A_p) \to L_p(B;\mathbf{R})$, such that for fixed $u \in \mathcal{D}(A_p)$ the map $t \to V(t)u$ is in $L_2([0,T];L_p(B;\mathbf{R}))$. There exists a constant c, dependent only on p and T, such that for all g_j as in Lemma 5.1 and all $t \in [0,T]$*

$$\int_B\int_\Omega \left|\sum_{k=1}^{j}\int_0^t [V(t-s)g_j^k(s,\omega)](x)\,dw_s^k\right|^p\ dP(\omega)\,d\Lambda(x)$$
$$\leq\quad c\int_B\int_\Omega\left(\int_0^t |[V(t-s)g_j(s,\omega)](x)|_{l_2}^2\,ds\right)^{\frac{p}{2}}\ dP(\omega)\,d\Lambda(x).$$

Proof. First fix some $t \in (0,T]$. For $x \in B$, $r > 0$ we define

$$Y_j(r,\omega,x) = \sum_{k=1}^{j}\int_0^r [V(t-s)g_j^k(s,\omega)](x)\,dw_s^k.$$

By the elementary structure of g_j,

$$\int_0^r \left|[V(t-s)g_j^k(s,\omega)](x)\right|^2\ ds < \infty$$

for allmost all $x \in B$, so that $Y_j(r,\omega,x)$ is well-defined as an Ito integral for such x, and it is a martingale. Since the Wiener processes w_s^k are independent, the quadratic variation of $Y_j(\cdot,\cdot,x)$ is

$$\sum_{k=1}^{j}\int_0^r \left|[V(t-s)g_j^k(s,\omega)](x)\right|^2\ ds.$$

Now the Burkholder-Davis-Gundy inequality (see [16, p. 163]) yields for $r \in [0,t]$,

$$\begin{aligned} &E\Big|\sum_{k=1}^{j}\int_0^r [V(t-s)g_j^k(s,\omega)](x)\ dw_s^k\Big|^p \leq \\ &cE\left(\int_0^r \sum_{k=1}^{j}|[V(t-s)g_j^k(s,\omega)](x)|^2\ ds\right)^{\frac{p}{2}} = \\ &cE\left(\int_0^r |V(t-s)g_j(s,\omega)](x)|_{l_2}^2\ ds\right)^{\frac{p}{2}}. \end{aligned} \tag{5.2}$$

In (5.2), take $r = t$ and integrate over B:

$$\int_B E\Big|\sum_{k=1}^{j}\int_0^t [V(t-s)g_j^k(s,\omega)](x)\ dw_s^k\Big|^p\,d\Lambda(x)\ \leq$$
$$c\int_B E\left(\int_0^t |V(t-s)g_j(s,\omega)](x)|_{l_2}^2\ ds\right)^{\frac{p}{2}}\ d\Lambda(x).$$

□

As a first application of the Lemma above we obtain that the series in (4.11) converges in $L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R}))$.

Lemma 5.3. *Assume that Hypotheses 4.1 and 4.2 hold. Let p, g, k_2, B be as in Theorem 4.3, with $\beta \in (\frac{1}{2}, 2)$. Take $\{\, g_j \,\}_{j=1}^{\infty}$ approximating g as in Lemma 5.1. Let $\eta > 0$ and asume that $\beta - \eta > \frac{1}{2}$. Then*

$$D_t^{\eta} \int_0^t k_2(t-s)\Sigma_{k=1}^{j} g_j^k(s,\omega,x)\; dw_s^k$$

converges in $L_p\big((0,T)\times\Omega;\mathcal{P};L_p(B;l_2)\big)$, when $j\to\infty$.

Proof. First note that under the assumption $\beta - \eta > \frac{1}{2}$ one has that $D_t^{\eta}(k_2 * f) = k * f$, where $k \in L_2(0,T)$. We work with this latter representation. In Lemma 5.2, take $V(t) = k(t)$ and integrate with respect to t to obtain

$$\begin{aligned}
&\int_0^T \int_B \int_\Omega \left| \sum_{k=1}^{j} \int_0^t k(t-s) g_j^k(s,\omega,x)\, dw_s^k \right|^p dP(\omega)\, d\Lambda(x)\, dt \;\le \\
&c \int_0^T \int_B \int_\Omega \left(\int_0^t |k(t-s) g_j(s,\omega,x)|_{l_2}^2 \; ds \right)^{\frac{p}{2}} dP(\omega)\, d\Lambda(x)\, dt \;= \\
&c \int_0^T \int_B \int_\Omega \left(\int_0^t |k(t-s)|^2 |g_j(s,\omega,x)|_{l_2}^2 \; ds \right)^{\frac{p}{2}} dP(\omega)\, d\Lambda(x)\, dt \;\le \\
&c \int_0^T \int_B \int_\Omega |g_j(t,\omega,x)|_{l_2}^p \, dP(\omega)\, d\Lambda(x)\, dt.
\end{aligned}$$

where we used $k^2 \in L_1(0,T)$ and the fact that

$$\left| k^2 * |g_j|_{l_2}^2 \right|_{L_{\frac{p}{2}}((0,T)\times\Omega\times B)} \le |k^2|_{L_1(0,T)} \left| |g_j|_{l_2}^2 \right|_{L_{\frac{p}{2}}((0,T)\times\Omega\times B)}.$$

Now recall that $g_j \to g$ in $L_p(\mathbf{R}_+ \times \Omega;\mathcal{P};L_p(B;l_2))$. □

Our solutions will be constructed by a stochastic variation-of-parameters formula using the (deterministic) resolvent associated with the triple (k_1, k_2, A_p). The resolvent theory for integral equations of evolutionary type is well understood. For the theory in case $\beta = 1$, see [23]. (See also [7]). For β not necessarily equal to 1, we define

Definition 5.4.

$$S_{\alpha\beta}(t)v \stackrel{\text{def}}{=} (2\pi i)^{-1} \int_{\Gamma_{1,\psi}} e^{\lambda t}(\lambda^{\alpha} I - A_p)^{-1}\lambda^{\alpha-\beta} v \; d\lambda, \quad t > 0, \tag{5.3}$$

for $v \in X$; where X is either $L_p(B;\mathbf{R})$ or $L_p(B;l_2)$, $\psi \in \big(\frac{\pi}{2}, \min\{\, \pi, \frac{\pi - \phi_{(-A_p)}}{\alpha} \,\}\big)$, and

$$\Gamma_{r,\psi} \stackrel{\text{def}}{=} \{\, re^{it} \mid |t| \le \psi \,\} \cup \{\, \rho e^{i\psi} \mid r < \rho < \infty \,\} \cup \{\, \rho e^{-i\psi} \mid r < \rho < \infty \,\}. \tag{5.4}$$

Lemma 5.5. *Let $\alpha \in (0,2)$, $\beta \in (\frac{1}{2},2)$, $\theta \in [0,1]$, $\eta \in (-1,1)$. Let $S_{\alpha\beta}(t)$ be the resolvent defined in Definition 5.4. Then one has*

$$S_{\alpha\beta}(t) \in \mathcal{L}(X),\ t>0;\quad \sup_{t>0}\|t^{1-\beta}S_{\alpha\beta}(t)\|_{\mathcal{L}(X)} < \infty, \tag{5.5}$$

$$S_{\alpha\beta}(t)v \in \mathcal{D}(A_p),\ t>0,\ v\in X;\quad \sup_{t>0}\|t^{1+\alpha-\beta}A_pS_{\alpha\beta}(t)\|_{\mathcal{L}(X)} < \infty, \tag{5.6}$$

$$\sup_{t>0}\|t^{1+\alpha\theta-\beta+\eta}D_t^{\eta}(-A_p)^{\theta}S_{\alpha\beta}(t)\|_{\mathcal{L}(X)} < \infty, \tag{5.7}$$

$$S_{\alpha\beta}(t) - A_p\int_0^t k_1(t-s)S_{\alpha\beta}(s)\ ds = k_2(t)I,\ t>0, \tag{5.8}$$

$$S_{\alpha\beta}(t) \textit{ is analytic for } t\in\mathbf{C},\ t\neq 0,\ |\arg\ t| < \psi - \frac{\pi}{2}. \tag{5.9}$$

Proof. To obtain (5.7), use (5.3), the analyticity of the integral and a change of variables to get

$$\begin{aligned}
&D_t^{\eta}(-A_p)^{\theta}S_{\alpha\beta}(t)\\
&= (2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^{\lambda t}(-A_p)^{\theta}(\lambda^{\alpha}I - A_p)^{-1}\lambda^{\alpha-\beta+\eta}\ d\lambda\\
&= (2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^{s}\big(\frac{s}{t}\big)^{\theta\alpha-\beta+\eta}t^{-1}(-A_p)^{\theta}\big(\frac{s}{t}\big)^{\alpha(1-\theta)}\big[(\frac{s}{t})^{\alpha} - A_p\big]^{-1}\ ds\\
&= ct^{-\theta\alpha+\beta-\eta-1}\int_{\Gamma_{1,\psi}} e^{s}s^{\theta\alpha-\beta+\eta}(-A_p)^{\theta}\big(\frac{s}{t}\big)^{\alpha(1-\theta)}\big[(\frac{s}{t})^{\alpha} - A_p\big]^{-1}\ ds.
\end{aligned} \tag{5.10}$$

Now apply (3.6) with $\mu = (\frac{s}{t})^{\alpha}$ to obtain that the last integral in (5.10) is bounded in $\mathcal{L}(X)$, uniformly in t. Estimates (5.5) and (5.6) are obtained similarly. Identity (5.8) follows by a straightforward Laplace transform argument, and the analyticity of $S_{\alpha\beta}$ is a consequence of its integral representation and Hypothesis 4.2. □

The central estimate in the proof is the following inequality:

Lemma 5.6. *Let $\alpha \in (0,2)$, $\beta \in (\frac{1}{2},2)$, $\theta \in [0,1]$, and $\eta \in (-1,1)$ such that (4.7) holds, i.e., $\beta - \alpha\theta - \eta > \frac{1}{2}$. Then there exists a constant c, depending on T, p, A, α, β, θ, η, such that for all $h \in L_p([0,T]\times B, l_2)$*

$$\begin{aligned}
&\int_0^T\int_B\Big(\int_0^t |D_t^{\eta}\big((-A_p)^{\theta}S_{\alpha\beta}(t-s)\big)h(s,x)|_{l_2}^2\ ds\Big)^{\frac{p}{2}}d\Lambda\ dt\\
&\le c\int_0^T\int_B |h(s,x)|_{l_2}^p\ d\Lambda\ ds.
\end{aligned} \tag{5.11}$$

Proof. Write $G(t) \stackrel{\text{def}}{=} D_t^{\eta}\big((-A_p)^{\theta}S_{\alpha\beta}\big)(t)$. First assume that $p > 2$. Then note that $\frac{p}{2}$, $\frac{p}{p-2}$ are conjugate exponents and let $f : [0,T]\times B \to \mathbf{R}_+$ be such that

$\int_0^T \int_B f^{\frac{p}{p-2}}\, d\Lambda\, dt = 1$. We estimate:

$$
\begin{aligned}
&\int_0^T \int_B f(t,x) \int_0^t |G(t-s)h(s,x)|_{l_2}^2 \; ds \; d\Lambda \; dt \\
&= \int_0^T \int_0^t \int_B f(t,x) |G(t-s)h(s,x)|_{l_2}^2 \; d\Lambda \; ds \; dt \\
&\le \int_0^T \int_0^t \Big[\int_B |f(t,x)|^{\frac{p}{p-2}} d\Lambda \Big]^{\frac{p-2}{p}} \Big[\int_B |G(t-s)h(s,x)|_{l_2}^p \; d\Lambda \Big]^{\frac{2}{p}} \; ds \; dt \\
&\le \int_0^T \int_0^t \|f(t,\cdot)\|_{L_{\frac{p}{p-2}}(B)} \|G(t-s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \|h(s,\cdot)\|^2_{L_p(B;l_2)} \; ds \; dt \\
&\le \Big[\int_0^T \|f(t,\cdot)\|^{\frac{p}{p-2}}_{L_{\frac{p}{p-2}}(B)} dt \Big]^{\frac{p-2}{p}} \\
&\qquad \Big[\int_0^T \Big| \int_0^t \|G(t-s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \|h(s,\cdot)\|^2_{L_p(B;l_2)} ds \Big|^{\frac{p}{2}} dt \Big]^{\frac{2}{p}},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\Big[\int_0^T \int_B \Big(\int_0^t |G(t-s)h(s,x)|_{l_2}^2 \; ds \Big)^{\frac{p}{2}} d\Lambda \, dt \Big]^{\frac{2}{p}} \\
&\le \Big[\int_0^T \Big| \int_0^t \|G(t-s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \; \|h(s,\cdot)\|^2_{L_p(B;l_2)} \; ds \Big|^{\frac{p}{2}} \; dt \Big]^{\frac{2}{p}}.
\end{aligned} \tag{5.12}
$$

In the case $p = 2$, equation (5.12) is obvious. Now (by convolution with respect to s) we obtain in either case

$$
\begin{aligned}
&\Big[\int_0^T \int_B \Big(\int_0^t |G(t-s)h(s,x)|_{l_2}^2 \; ds \Big)^{\frac{p}{2}} d\Lambda \, dt \Big]^{\frac{2}{p}} \\
&\le \Big[\int_0^T \|G(t)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \, dt \Big] \; \Big[\int_0^T \|h(t,x)\|^p_{L_p(B;l_2)} \, dt \Big]^{\frac{2}{p}} \\
&\le c \, \|h(t,x)\|^2_{L_p((0,T);L_p(B;l_2))},
\end{aligned}
$$

where the last inequality follows by (5.7) and (4.7). Thus (5.11) holds. ☐

Remark 5.7. The constant c in the Lemma above can be made arbitrarily small by choosing a sufficiently short time interval $[0, T]$.

Proof. In fact, the last lines of the proof above show that c is proportional to $\int_0^T \|G(t)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \, dt$, which converges to 0 as $T \to 0$. ☐

We have now collected all tools for the proof of Theorem 4.3. To proceed, we use Lemma 5.1 and therefore consider

$$
\begin{aligned}
u_j(t,\omega,x) \;&=\; \int_0^t k_1(t-s) A_p u_j(s,\omega,x) \; ds \\
&\qquad + \sum_{k=1}^{j} \int_0^t k_2(t-s) g_j^k(s,\omega,x) \; dw_s^k,
\end{aligned} \tag{5.13}
$$

where the functions $g_j = \{\, g_j^k \,\}_{k=1}^j$ are such that

$$
\|g_j - g\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B;l_2))} \to 0 \tag{5.14}
$$

for $j \to \infty$, and where each g_j^k is of the simple structure given by Lemma 5.1. We construct first solutions to (5.13), and then show that these converge.

For fixed j, let u_j be defined by

$$u_j(t,\omega,x) = \sum_{k=1}^{j} \int_0^t S_{\alpha\beta}(t-s) g_j^k(s,\omega,x)\ dw_s^k. \tag{5.15}$$

To see that the stochastic integrals in (5.15) are well-defined, notice that for each fixed $v \in L_p(B;l_2)$, $S_{\alpha\beta}(s)v$ is smooth in time for $s>0$ by (5.9). Since g_j is of the simple form (5.1), it is easy to see that $S_{\alpha\beta}(t-s)g_j(s,\omega,x)$ is adapted. From $\beta > \frac{1}{2}$ and (5.5) we have that $S_{\alpha\beta}(s)v$ is in $L_2([0,T];L_p(B;l_2))$.

Note that by (5.6), $u_j \in \mathcal{D}(A_p)$ for $t>0$, a.s. In fact we have more: Since g_j^k takes values in $G \subset \mathcal{D}(A_p)$, and since A_p and $S_{\alpha\beta}(t)$ commute on $\mathcal{D}(A_p)$,

$$A_p u_j(t,\omega,x) = \sum_{k=1}^{j} \int_0^t S_{\alpha\beta}(t-s) A_p g_j^k(s,\omega,x)\ dw_s^k. \tag{5.16}$$

We next claim that u_j defined by (5.15) satisfies (5.13) for $t>0$, a.e. on B, and a.s. (For simplicity, without loss of generality, take $j=1$, and $g_1^1 = I_{\tau_1<s\le\tau_2}(s)g(x)$). We have

$$\begin{aligned}
k_1 * A_p u_j &= \int_0^t k_1(t-s)[\int_0^s S_{\alpha\beta}(s-\tau) I_{\tau_1<\tau\le\tau_2}(\tau) A_p g(x)\ dw_\tau\ ]ds \\
&= \int_0^t \Big[\int_\tau^t k_1(t-s) S_{\alpha\beta}(s-\tau) A_p g(x)\ ds\Big] I_{\tau_1<\tau\le\tau_2}\ dw_\tau \\
&= \int_0^t \Big[\int_0^{t-\tau} k_1(t-\tau-v) S_{\alpha\beta}(v) A_p g(x)\ dv\Big] I_{\tau_1<\tau\le\tau_2}\ dw_\tau \\
&= \int_0^t \Big[A_p \int_0^{t-\tau} k_1(t-\tau-v) S_{\alpha\beta}(v) g(x)\ dv\Big] I_{\tau_1<\tau\le\tau_2}\ dw_\tau \\
&= \int_0^t \Big[S_{\alpha\beta}(t-\tau) - \frac{1}{\Gamma(\beta)}(t-\tau)^{\beta-1}\Big] g(x) I_{\tau_1<\tau\le\tau_2}\ dw_\tau \\
&= u_j(t,\omega,x) - \int_0^t k_2(t-\tau) g_1^1(\tau,\omega,x)\ dw_\tau.
\end{aligned} \tag{5.17}$$

The first equality in (5.17) follows by (5.16). The second is a consequence of the stochastic Fubini theorem. For this, see, e.g. [26, Th.4.6, p.160] where we take $d\mu(s) = k_1(t-s)ds$ and observe that $\int_0^t \int_0^s \|S_{\alpha\beta}(s-\tau)\|^2_{X\to X} k_1(t-s)\ d\tau ds < \infty$, by the fact that $\alpha>0, \beta>\frac{1}{2}$ and (5.5). The third equality is a simple change of variables. The fourth follows by the commutativity of $S_{\alpha\beta}$, A_p on $\mathcal{D}(A_p)$. The next to last is (5.8). The last equality uses (5.15).

To obtain apriori bounds for the approximating solutions, let $\theta \in [0,1]$ and $\eta < 1$ be such that (4.7) holds, i.e. $\beta - \alpha\theta - \eta > \frac{1}{2}$. For shorthand put $V(t) = D_t^\eta (-A)^\theta S_{\alpha\beta}(t)$. For $t>0$, $V(t)$ is a bounded linear operator. By definition of D_t^η we have that for all $h \in L_p(B;\mathbf{R})$

$$(-A)^\theta S_{\alpha\beta}(t)h = \int_0^t \frac{\tau^{\eta-1}}{\Gamma(\eta)} V(t-\tau)h\, d\tau.$$

Consequently, using the stochastic Fubini theorem, we have

$$\begin{aligned}(-A)^\theta u_j(t,\omega,x) &= \sum_{k=1}^{j}\int_0^t [(-A)^\theta S_{\alpha\beta}(t-s)g_j^k(s,\omega)](x)\,dw_s^k\\ &= \sum_{k=1}^{j}\int_0^t\int_0^{t-s}\frac{\tau^{\eta-1}}{\Gamma(\eta)}\,[V(t-s-\tau)g_j^k(s,\omega)](x)\,d\tau\,dw_s^k\\ &= \int_0^t \frac{\tau^{\eta-1}}{\Gamma(\eta)}\,\big(\sum_{k=1}^{j}\int_0^{t-\tau}[V(t-\tau-s)g_j^k(s,\omega)](x)\,dw_s^k\big)\,d\tau.\end{aligned}$$

This shows that

$$D_t^\eta(-A)^\theta u_j(t,\omega,x) = \sum_{k=1}^{j}\int_0^t [V(t-s)g_j^k(s,\omega)](x)\,dw_s^k.$$

We apply Lemma 5.2 and Lemma 5.6

$$\begin{aligned}&\int_0^T\int_B\int_\Omega \left|D_t^\eta(-A)^\theta u_j(t,\omega,x)\right|^p\,dP(\omega)\,d\Lambda(x)\,dt\\ &= \int_0^T\int_B\int_\Omega \left|\sum_{k=1}^{j}\int_0^t [V(t-s)g_j^k(s,\omega)](x)\,dw_s^k\right|^p\,dP(\omega)\,d\Lambda(x)\,dt\\ &\le c\int_0^T\int_B\int_\Omega \left(\int_0^t |[V(t-s)g_j(s,\omega)](x)|_{l_2}^2\,ds\right)^{\frac{p}{2}}\,dP(\omega)\,d\Lambda(x)\,dt\\ &\le c\int_0^T\int_B\int_\Omega |g_j(s,\omega,x)|_{l_2}^p\,dP(\omega)\,d\Lambda(x)\,ds.\end{aligned}\tag{5.18}$$

Here the constant c depends on T, α, β, θ, η, A_p, and p, but not on g_j and j.

We next let $j\to\infty$. By linearity, by (5.14), and by (5.18),

$$\|u_j-u_i\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B))} \le c\|g_j-g_i\|_{L_p((0,T)\times\Omega;\mathcal{P};L_p(B,l_2))}\to 0$$

for $i,j\to\infty$. By completeness, there exists u such that $u_j\to u$ in $L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R}))$. Moreover, $(k_1*u_j)\in\mathcal{D}(A_p)$ and $k_1*u_j\to k_1*u$ in $L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R}))$. By Lemma 5.3, (take $\eta=0$), the series $k_2\star\Sigma_{k=1}^j g_j^k$ converges. Since $u_j = A_p(k_1*u_j)+k_2\star\Sigma_{k=1}^j g_j^k$, we have that $A_p(k_1*u_j)$ converges. By the closedness of A_p, $k_1*u\in\mathcal{D}(A_p)$ and (4.3) holds.

In case $\beta-\alpha\theta>\frac{1}{2}$, then by (5.18) with $\eta=0$, $(-A_p)^\theta u_j$ converges. The closedness of the fractional powers gives (4.5), (4.6). Analogously, by the closedness of D_t^η, and by (5.18), we have (4.8).

The relation (4.9) is a consequence of (4.7), (4.8), [30, p. 29]. For the application of the results in [30], note that here $u(t=0)=0$, and that if $\beta-\alpha\theta-\frac{1}{2}>1+\frac{1}{p}$, then $u'(t=0)$ is welldefined and $=0$.

Recalling Lemma 5.3, we conclude that (4.10) is satisfied.

Finally, to prove uniqueness, assume that u and $\tilde{u}$ are two solutions of (4.3). Then $v=u-\tilde{u}$ solves the deterministic integral equation

$$v = k_1*A_pv.$$

From the theory of deterministic integral equations [23] we know that $v=0$. Now Theorem 4.3 is proved.

Remark 5.8. Estimate (4.8) can be refined: Let $a\in L_1([0,T];\mathbf{R})$ for all $T<\infty$, and of subexponential growth. Define its Laplace transform $\hat{a}(s)=\int_0^\infty e^{-st}a(t)\,dt$.

Suppose that there are functions $f : (0,\infty) \to [0,\infty)$ and $g : \Gamma_{1,\psi} \to [0,\infty)$ (c.f. (5.4)) such that

$$|\hat{a}(\frac{s}{t})| \le f(t)g(s) \quad \text{for all } s \in \Gamma_{1,\psi},\ t \in (0,T],$$
$$t^{\beta-\alpha\theta-2} f(t) \in L_2((0,T);\mathbf{R}),$$
$$\int_{\Gamma_{1,\psi}} |e^s|\,|s|^{\alpha\theta-\beta+1} g(s)\,|ds| < \infty.$$

Then the solution u of (4.3) satisfies

$$\frac{d}{dt}(-A)^\theta a * u \in L_p((0,T)\times\Omega; L_p(B;\mathbf{R})).$$

In particular, this holds if

$$|\hat{a}(\lambda)| \le c\,|\lambda|^{\eta-1} \quad \text{with } \beta-\alpha\theta-\eta > \frac{1}{2}. \tag{5.19}$$

Proof. For shorthand we write

$$V(t) = \frac{d}{dt}(-A)^\theta[a * S_{\alpha\beta}](t).$$

Copying the proof of (5.7) we obtain

$$\|V(t)\|_{\mathcal{L}(X)} \le \frac{1}{2\pi} t^{\beta-\alpha\theta-2} f(t) \int_{\Gamma_{1,\psi}} e^s|s|^{\alpha\theta-\beta+1} g(s)\,|ds| \le ct^{\beta-\alpha\theta-2} f(t).$$

Thus $\|V(t)\|_{\mathcal{L}(X)}$ is in $L_2((0,T);\mathbf{R})$ and we can redo the proof of Lemma 5.6 to obtain that

$$\int_0^T \int_B \left(\int_0^t |V(t-s)h(s,x)|_{l_2}^2\,ds\right)^{\frac{p}{2}} d\Lambda\,dt$$
$$\le c\int_0^T \int_B |h(s,x)|_{l_2}^p\,d\Lambda\,ds.$$

Now we can again use Lemma 5.2 to achieve the desired result.
In particular, if (5.19) holds, take $f(t) = ct^{1-\eta}$ and $g(s) = |s|^{\eta-1}$. □

6. Proof of Theorem 4.10

Let u_j be approximate solutions as in the proof of Theorem 4.3, c.f., (5.15). By [25, Thm.1, p. 295] and the assumption on A_p, one has that the resolvent $S_{\alpha,1}(t)$ admits the kernel representation

$$(S_{\alpha,1}(t)f)(x) = \int_B \sigma_t(x,y)f(y)\,dy, \quad x \in B,\ t > 0,$$

for $f \in L_p(B;\mathbf{R})$. Thus

$$u_j(t,\omega,x) = \sum_{k=1}^{j} \int_0^t \Big[\int_B \sigma_{t-s}(x,y) g_j^k(s,\omega,y)\,dy\Big]\,dw_s^k.$$

The kernel σ_t satisfies the estimate

$$|\sigma_t(x,y)| \le t^{-\frac{\alpha n}{m}}\, q\left(t^{-\frac{\alpha}{m}}|x-y|\right), \text{ for } t > 0,$$

with a continuous, nonincreasing function $q : (0,\infty) \to (0,\infty)$ such that

$$\int_B t^{-\frac{\alpha n}{m}}\, q\left(t^{-\frac{\alpha}{m}}|x|\right)\,dx = \int_0^\infty q(r)r^{n-1}\,dr < \infty.$$

In particular (by convolution of an L_1-function with an L_2-function) we infer that there exists a constant c independent of t

$$\int_B \left| \int_B \sigma_t(x,y) g_j(s,\omega,y)\,dy \right|_{l_2}^2 dx \le c \int_B |g_j(s,\omega,x)|_{l_2}^2\,dx. \tag{6.1}$$

Using Lemma 5.2 and (6.1), we obtain

$$\begin{aligned}
&\int_\Omega \int_B |u_j(t,\omega,x)|^2\,dx\,dP(\omega)\\
&= \int_\Omega \int_B \left| \sum_{k=1}^{j} \int_0^t \int_B \sigma_{t-s}(x,y) g_j^k(s,\omega,y)\,dy\,dw_s^k \right|^2 dx\,dP(\omega)\\
&\le c \int_\Omega \int_B \int_0^t \left| \int_B \sigma_{t-s}(x,y) g_j^k(s,\omega,y)\,dy \right|_{l_2}^2 ds\,dx\,dP(\omega)\\
&\le c \int_\Omega \int_B \int_0^t |g_j(s,\omega,x)|_{l_2}^2\,ds\,dx\,dP(\omega)\\
&\le c\,\|g_j\|^2_{L_2([0,T]\times\Omega\times B;l_2)}.
\end{aligned}$$

Hence, for almost all ω, u_j converges in $L_\infty\big((0,T);L_2(B;\mathbf{R})\big)$. Since each $u_j \in C\big([0,T];L_2(B;\mathbf{R})\big)$, the claim follows.

7. Proof of Theorem 4.11

To prove (i), observe that by [23, Ch. I] and by (4.1), the resolvent $S(t) \stackrel{\text{def}}{=} S_{\alpha 1}(t)$ for (2.1) (exists and) is bounded, analytic. Thus $u_1 \stackrel{\text{def}}{=} S(t)u_0$ satisfies

$$u_1 = u_0 + A_p(k_1 * u_1), \quad t > 0,$$

and a.a. ω. In addition we have that $u_1 \to u_0$ as $t \downarrow 0$ [23, p. 32], Moreover, $u_1 \in \mathcal{D}(A_p)$ for $t > 0$. By transformation techniques, one has, for a.a. ω,

$$\|(-A_p)^\theta u_1(t,\omega,\cdot)\|_{L_p(B)} \le ct^{-\alpha\theta} \;;\quad \theta \in [0,1].$$

Therefore, if $\alpha\theta < 1$, $\int_0^t \|k_1(t-s)(-A_p)^\theta u_1(s)\|_{L_p(B)}\,ds < \infty$ and (4.25), (4.26) follow.

To obtain (4.27), note that by estimating the inverse Laplace transform, one has

$$\|D_t^\eta (u_1 - u_0)\|_{L_p(\Omega\times B)} \le c(\eta) t^{-\eta} \|u_0\|_{L_p(\Omega\times B)}.$$

For the proof of (4.29), one applies results of [8] as follows. Take E_0, E_1 of [8, p. 427], respectively, equal to $L_p(\Omega;L_p(B))$, $L_p(\Omega;\mathcal{D}(A_p))$ and observe that (X a Banach space)

$$\begin{aligned}
&BUC_{1-\mu}([0,T],X)\\
&\stackrel{\text{def}}{=} \{\, u \in C((0,T];X) \mid t^{1-\mu}u(t) \in BUC((0,T];X), \lim_{t\downarrow 0} t^{1-\mu}\|u(t)\|_X = 0 \,\}.
\end{aligned}$$

First use (21) of [8] which gives (by (4.28)) that

$$u_1 \in BUC_{1-\mu}\Big([0,T];L_p\big(\Omega;\mathcal{D}(A_p)\big)\Big) \cap BUC^\alpha_{1-\mu}\Big([0,T];L_p\big(\Omega;L_p(B)\big)\Big),$$

where (see [8, p. 428]) $\mu = 1-\alpha+\alpha\hat{\mu}$. Here, [8, p. 423],

$$BUC^\alpha_{1-\mu}\Big([0,T];L_p\big(\Omega;L_p(B;\mathbf{R})\big)\Big)$$

is the set of $u \in BUC_{1-\mu}\Big([0,T];L_p\big(\Omega;L_p(B;\mathbf{R})\big)\Big)$ for which there exist

$$u_0 \in L_p(B) \;\text{ and }\; f \in BUC_{1-\mu}\Big([0,T];L_p\big(\Omega;L_p(B;\mathbf{R})\big)\Big)$$

such that

$$u = u_0 + \frac{1}{\Gamma(\alpha)} t^{-1+\alpha} * f.$$

Observe that by the condition on $\hat{\mu}$ one has $\mu > 1 - p^{-1}$ which implies (see [8, p. 422])

$$BUC_{1-\mu}\big([0,T]; L_p(B;\mathbf{R})\big) \subset L_p\big((0,T); L_p(B;\mathbf{R})\big).$$

Thus

$$u_1 = u_0 + \frac{1}{\Gamma(\alpha)} t^{-1+\alpha} * f,$$

with $f \in L_p\big([0,T]; L_p(\Omega; L_p(B;\mathbf{R}))\big)$ and (4.29) follows.

To obtain (4.29) under the assumption (4.30), observe that (see, e.g., [19, p. 47 and p. 56])

$$\mathcal{D}\big((-A_p)^\theta\big) \subset \big(L_p(B), \mathcal{D}(-A_p)\big)_{\hat{\mu}},$$

for $\theta > \hat{\mu}$. For θ as in (4.30) we can find $\hat{\mu} < \theta$ such that $\hat{\mu} > 1 - \frac{1}{\alpha p}$.

8. Proof of Theorem 4.14

In fact, we can almost literally copy the proof of (4.8) (for the special case $\eta = 0$) from Theorem 4.3. There (4.8) is proved for the approximate solutions u_j (c.f. (5.15)) by the estimate (5.18), based on Lemma 5.6 and Lemma 5.2. By a straightforward limiting procedure the estimate is carried over to the solution u. It only is Lemma 5.6, where the full strength of condition (4.7) is required. In the case of Theorem 4.14, we can complete Lemma 5.6 by the following result, which works in the case that $\beta - \alpha\theta = \frac{1}{2}$. The rest of the proof can be copied from the proof of Theorem 4.3.

Lemma 8.1 ([11, Theorem 1.2])**.** *Let $n \geq 1$ be an integer, $\alpha \in (0,2)$, $\beta > \frac{1}{2}$, $\theta \in (0,1)$ such that $\beta - \alpha\theta = \frac{1}{2}$. Let H be a separable Hilbert space (e.g., $H = l_2$), $p \in [2,\infty)$ and $T \in \mathbf{R}$. Let $S_{\alpha\beta}$ be the resolvent operator corresponding to* (4.32)*. Then there exists some constant c (depending on p, α, β, θ, n) such that for all $h \in L_p((-\infty,T)\times\mathbf{R}^n; H)$ the following estimate holds:*

$$\begin{aligned} &\int_{\mathbf{R}^n}\int_{-\infty}^{T}\left[\int_{-\infty}^{t} \left\|\left[(-\Delta)^\theta S_{\alpha\beta}(t-s)h(s,\cdot)\right](x)\right\|_H^2 ds\right]^{\frac{p}{2}} dt\,dx \\ &\leq c\int_{\mathbf{R}^n}\int_{-\infty}^{T} \|h(s,y)\|_H^p\,ds\,dy. \end{aligned} \tag{8.1}$$

This lemma is a generalization of [17, Theorem 2.1]. Its proof (and thus closing the gap to obtain maximal regularity) is significantly more intricate than proving Lemma 5.6.

9. Examples

Our first example is well-known. See, e.g., [22].

Let $B \subset \mathbf{R}^n$ be a bounded domain with C^2-boundary. Assume $s > n$, $s \geq p$, $p \in [2,\infty)$. Let $b \in W^{1,s}(B,\mathbf{R}^{n\times n})$ and assume b is symmetric and uniformly positive definite. Assume A_p is a second order operator in divergence form, i.e.,

$$(A_p u)(x) \overset{\text{def}}{=} \operatorname{div}(b(x)\cdot\nabla u(x)), \quad x \in B,$$

for $u \in \mathcal{D}(A_p) = W^{2,p}(B) \cap W_0^{1,p}(B)$.

Then A_p is closed, densely defined, positive in $L_p(B)$ with compact resolvent and

$$\|(\lambda I - A_p)^{-1}\| \leq \frac{c(\psi)}{1+|\lambda|}; \quad |\arg\ \lambda| \leq \pi - \psi,$$

where $\psi > 0$ arbitrary. Hence $\phi_{-A_p} < \pi(1-\frac{\alpha}{2})$ for any $\alpha \in (0,2)$, and so Theorem 4.3 may be applied. In particular, with $\alpha \in (0,2)$, $\beta \in (\frac{1}{2},2)$, and g satisfying (4.2), we have that there exists a unique solution $u \in L_p((0,T)\times\Omega;\mathcal{P};L_p(B;\mathbf{R}))$ such that $k_1 * u \in \mathcal{D}(A_p)$ for $t>0$ and $a.a.\ \omega \in \Omega$, and such that (4.3) holds.

If, in addition, $\beta - \alpha > \frac{1}{2}$, then $u \in \mathcal{D}(A_p)$, for $t > 0$ and $a.a.\ \omega \in \Omega$, and

$$u = k_1 * A_p u + \sum_{k=1}^{\infty} k_2 \star g^k. \tag{9.1}$$

Our second example concerns the case where A_p is not of divergence form. We make use of [24].

Assume $B \subset \mathbf{R}^n$ and, e.g., either $B = \mathbf{R}^n$ or B a bounded domain with C^2-boundary. Let $p \in [2,\infty)$, and

$$\tilde{A}_p u = \sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k u(x) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u(x)$$

for $u \in \mathcal{D}(A_p) = W^{2,p}(B) \cap W_0^{1,p}(B)$. Here $a(x) = (a_{jk}(x))$ is a real symmetric matrix for $x \in \overline{B}$ such that

$$0 < a_0 \leq a(x)\xi\cdot\xi \leq \frac{1}{a_0}, \quad for \quad x \in \overline{B},\ |\xi| = 1.$$

Assume $a_{ij} \in C^\rho(\overline{B})$, for some $\rho \in (0,1)$; and if B is unbounded, assume $a_{ij}^\infty = \lim_{|x|\to\infty} a_{ij}(x)$ exists, with $|a_{ij}(x) - a_{ij}^\infty| \leq c|x|^{-\rho}$, for $|x|$ large and all i,j. Then, under certain conditions on b_j, c (see [24, (A3), (A4), p. 165]), the spectrum of $-\tilde{A}_p$ away from $[0,\infty)$ consists only of eigenvalues.

In particular, assume that ψ is such that $\tilde{A}_p$ has no eigenvalues λ such that $0 \neq \lambda \in \overline{\Sigma}_{\pi-\psi}$. Then, [24, Th. D(c)], for each $\eta > 0$ there exists $c_p(\eta) > 0$ such that

$$\|(\lambda I - \tilde{A}_p)^{-1}\| \leq c_p(\eta)|\lambda|^{-1}, \quad \lambda \in \Sigma_{\pi-\psi}, \quad |\lambda| > \eta.$$

Thus, Theorem 4.3 may be applied with $A_p = \tilde{A}_p - \delta I$, for any $\delta > 0$, provided

$$0 < \alpha < 2\big[1 - \frac{\psi}{\pi}\big], \quad \beta > \frac{1}{2},$$

to give a solution u satisfying (4.3). If

$$0 < \alpha < \min\big(\beta - \frac{1}{2}, 2[1-\frac{\psi}{\pi}]\big),$$

we have a solution u satisfying (9.1).

Note moreover that (under the above assumptions) we have for any $\delta > 0$, that $\delta - \tilde{A}_p$ admits bounded imaginary powers [24, Th.D(c)]. In addition, if B is a bounded domain and $\mathcal{N}(\tilde{A}_p) = 0$, then, [24, Th. D(c) and Th. A],

$$\|(-\tilde{A}_p)^{iy}\| \leq ce^{\theta|y|}, \quad y \in \mathbf{R}.$$

Hence we have, in this case, $\mathcal{D}\big((-\tilde{A}_p)^\theta\big) = \big[L_p(B), \mathcal{D}(\tilde{A}_p)\big]_\theta$.

As a final remark we point out the following.

Suppose $(-A_p)$ has spectral angle $\omega \stackrel{\text{def}}{=} \phi_{-A_p},$. Take $\theta\omega < \pi$. Recall that then $(-A_p)^\theta$ has spectral angle $\theta\omega$. Thus the above examples may be extended, e.g., to fourth order operators A obtained as $A = (-A_p)^2$, under appropriate conditions on the spectral angle of $-A_p$.

10. Krylov's Approach Versus B-space Valued Stochastic Integration

At the center of the study of stochastic integral equations in Banach spaces is the problem of defining and estimating stochastic integrals, in particular stochastic convolutions, in Banach spaces. Krylov's approach, which is used in this paper, is elementary in the sense that stochastic integrals are taken pointwise, so they are classical Ito-integrals of scalar valued processes. Kahane-Khintchine arguments, in particular the Burkholder-Davis-Gundy inequality, provide the step from L_2-estimates to L_p. Of course, this can only be done for sufficiently "nice" integrands. The final step is to extend the results obtained for smooth initial data and elementary forcing terms to more general L_p-data by a completion argument.

On the other hand, the recent progress on stochastic integration in Banach spaces (see, e.g.[21]) provides a convenient tool to handle stochastic convolutions directly in the Banach space. While we do not know about applications of this method to integral equations, it has been used successfully to treat parabolic stochastic differential equations, e.g., [12], [29]. We expect that such results can be extended to integral equations. Clearly, this approach works in more general Banach spaces, while the more classical technique is confined to the special structure of L_p.

It seems interesting to conclude our paper with a short comparison of these two approaches.

We mention first, that the results obtained for parabolic differential equations go far beyond the scope of our paper, in the sense that both approaches have been used to treat nonlinear equations, with possibly time dependent coefficients, and state dependent diffusion. On the other hand, the aim of the present paper is to treat a fractional differential equation. With $\alpha = \beta = 1$, our equation (1.1) reduces to the stochastic differential equation

$$du(t) = A_p u(t)\, dt + G(t)\, dW_t. \tag{10.1}$$

It is this case, where we can compare our results to the results obtained by the abstract integration theory. Notice that in abstract notation, W_t is a cylindrical Wiener process in a separable Hilbert space H and $G \in L_p([0,T] \times \Omega; \gamma(H, L_p(B))$ where $\gamma(H, L_p(B))$ denotes the space of γ-radonifying operators $H \to L_p(B)$. This is equivalent to writing the stochastic forcing in Krylov's notation

$$g(t) = \sum_{k=1}^{\infty} g^k w_s^k.$$

with $\{g^k\}_{k=1}^{\infty} \in L_p([0,T] \times \Omega; L_p(B, l_2))$ (use, e.g., [29, Proposition 3.2.3]).

With the assumption that A is sectorial, [29, Theorem 8.2.1] states (translated to our notation, and in the special case $\theta = 0$ and time-independent generators $A(t) = A$)

$$\|u\|^p_{L_p([0,T]\times\Omega;(E,\mathcal{D}(A))_{\delta,1})} \leq c\|G\|^p_{L_p([0,T]\times\Omega;\gamma(H,E))} \quad \text{if } \delta < \frac{1}{2},$$
$$u \in C^{\lambda}([0,T], (E, \mathcal{D}(A))_{\delta,1}) \text{ a.e.} \quad \text{if } \delta + \lambda < \frac{1}{2} - \frac{1}{p}.$$

These results are exactly the results of our Theorem 4.3(c) and Corollary 4.8(c) with $E = L_p(B)$, $\alpha = \beta = 1$, $\eta = 0$, $\theta = \delta$. (Notice that the condition on δ is a strict inequality, so that it makes no difference whether the result is stated in terms of $(E, \mathcal{D}(A))_{\delta,1}$ or of $\mathcal{D}((-A)^{\delta})$. In [12, Theorem 4.1(ii)] we find that $u \in C^{\lambda}([0,T], \mathcal{D}(-A)^{\theta})$ if $\theta + \lambda < \frac{1}{2}$. Notice that in this result the forcing operator $G(t)$ is independent of time, so it is in $L_p([0,T] \times \Omega, E)$ for all $p \geq 1$, and the result is again commensurate to Corollary 4.8 in the present paper.

In the abstract approach, maximal regularity is achieved in the case that A has a γ-bounded H^∞-calculus. In this case, and Hilbert space, the maximal regularity result [29, Theorem 8.2.2] (again for the case of time-independent generators) reads

$$\|u\|^2_{L_2([0,T]\times\Omega;[E,D]_{1/2})} \leq c\|G\|^2_{L_2([0,T]\times\Omega;\gamma(H,E))}$$

which is exactly what we get in Theorem 4.14, if A is the Laplacian on $\mathbf{R}^n$, $p = 2$ and $\theta = \frac{1}{2}$. Theorem 4.14 is not confined to the Hilbert space case. A comparable maximal regularity result for a space which needs only be of finite cotype, and again with γ-bounded H^∞-calculus is given in [12, Theorem 6.2]. Unlike our Theorem 4.14, it is pointwise for $t \in [0, T]$, but again it is based on the assumption that G is independent of time. Again, both approaches are equivalent for the Laplacian. Notice, however, that the proof of Theorem 4.14 is based on the maximal inequality Lemma 8.1, which is derived only for the Laplacian operator on the full space, and that the proof of this lemma is quite sophisticated and takes a paper of its own [11].

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