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AMS subject classifications: 65F10, 15A18

Keywords: $\mathbb{R}$-linear GMRES, equivalent real formulation, spectral analysis

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MINIMAL RESIDUAL METHODS FOR SOLVING A CLASS OF $\mathbb{R}$-LINEAR SYSTEMS OF EQUATIONS

KUI DU[†] AND OLAVI NEVANLINNA[†]

Abstract. Recently, $\mathbb{R}$-linear GMRES was proposed by Eirola, Huhtanen and von Pfaler [*SIAM J. Matrix Anal. Appl.* 25 (2004), pp. 804-828] for solving a class of $\mathbb{R}$-linear systems of equations. In this work we investigate $\mathbb{R}$-linear GMRES through the equivalent real formulations of the $\mathbb{R}$-linear system. We show that $\mathbb{R}$-linear GMRES requires fewer matrix-vector products than GMRES applied to the related $\mathbb{C}$-linear system. Numerical results for an artificial example and the inverse problem of reconstructing an unknown electric conductivity are reported to confirm our theoretical results.

Key words. $\mathbb{R}$-linear GMRES, equivalent real formulation, spectral analysis

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1. Introduction. Consider the $\mathbb{R}$-linear system

$$\kappa z + M\bar{z} = b, \tag{1.1}$$

where $\kappa \in \mathbb{C}, z, b \in \mathbb{C}^n, M \in \mathbb{C}^{n\times n}$ and $\bar{z}$ denotes the conjugate of z. We use $\mathcal{M}_\kappa z = (\kappa I + M\tau)z$ to denote the left-hand side, where I denotes the identity matrix whose dimension is clear from the context and $\tau z = \bar{z}$ is the conjugation operator on $\mathbb{C}^n$. Such $\mathbb{R}$-linear systems arise in the inverse problem of reconstructing an unknown electric conductivity in the unit disc from boundary measurements [18, 14, 1], especially in the numerical discretization of the $\mathbb{R}$-linear Beltrami equation [11] and the $\bar{\partial}$-equation [13]. For theoretical analysis of general $\mathbb{R}$-linear operators $\mathcal{A} = M + N\tau$ where $M, N \in \mathbb{C}^{n\times n}$, we refer to [6, 9, 10, 12].

By rewriting (1.1) as an equivalent real formulation of doubled size for its real and imaginary parts, any standard Krylov subspace method [16, 21] can de used to solve the problem. Equivalent real formulations of a $\mathbb{C}$-linear (complex linear) system have been considered by several researchers, see, for example, [7, 2, 4, 3]. Recently, the $\mathbb{R}$-linear GMRES (RL-GMRES) method [6] was proposed for solving (1.1), which avoids using an equivalent real formulation. In this paper, we investigate RL-GMRES through the equivalent real formulation of (1.1).

Huhtanen and Perämäki [11] considered preconditioning techniques for the $\mathbb{R}$-linear system (1.1). By the right preconditioner $\bar{\kappa}I - M\tau$, they obtained the $\mathbb{C}$-linear system

$$|\kappa|^2 w - M\overline{M}w = b. \tag{1.2}$$

If (1.2) is solved, then z can be readily obtained by $z = \bar{\kappa}w - M\bar{w}$. In this paper, we show that RL-GMRES applied to (1.1) is faster than GMRES [17] applied to (1.2) in terms of matrix-vector products, namely, RL-GMRES requires fewer matrix-vector products; see Remark 3.10.

The paper is organized as follows. In section 2 we discuss equivalent real formulations of (1.1). Spectral properties of the equivalent real formulations of the $\mathbb{R}$-linear operator $\mathcal{M}_\kappa$ are studied. In section 3 we review RL-GMRES and investigate it through the equivalent real formulation. In section 4 we report numerical experiments confirming our theoretical results. In the last section we present our conclusions.

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2. Equivalent real formulations. Write $\kappa = \alpha + \mathrm{i}\beta$, $z = x + \mathrm{i}y$, $b = c + \mathrm{i}d$ and $M = A + \mathrm{i}B$, where $\mathrm{i} = \sqrt{-1}$, $\alpha, \beta \in \mathbb{R}$, $x, y, c, d \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$. The $\mathbb{R}$-linear system (1.1) admits several $2n \times 2n$ equivalent real formulations. We list four representative ones in (2.1)-(2.4), and call them R1 to R4, respectively.

R1 formulation.

$$\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix} + \alpha \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \beta \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}. \tag{2.1}$$

R2 formulation.

$$\left(\begin{bmatrix} A & B \\ B & -A \end{bmatrix} + \alpha \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \beta \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}. \tag{2.2}$$

R3 formulation.

$$\left(\begin{bmatrix} B & A \\ A & -B \end{bmatrix} + \alpha \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} + \beta \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} d \\ c \end{bmatrix}. \tag{2.3}$$

R4 formulation.

$$\left(\begin{bmatrix} B & -A \\ A & B \end{bmatrix} + \alpha \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \beta \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d \\ c \end{bmatrix}. \tag{2.4}$$

For notational simplicity, let

$$\mathbf{M}_1 = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}, \mathbf{M}_2 = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}, \mathbf{M}_3 = \begin{bmatrix} B & A \\ A & -B \end{bmatrix}, \mathbf{M}_4 = \begin{bmatrix} B & -A \\ A & B \end{bmatrix}, \tag{2.5}$$

and

$$\mathbf{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{2.6}$$

Let $\|\cdot\|$ denote the Euclidean vector norm or the associated induced matrix norm. We have

$$\|\mathbf{M}_1\| = \|\mathbf{M}_2\| = \|\mathbf{M}_3\| = \|\mathbf{M}_4\| = \|M\|. \tag{2.7}$$

Let $\mathbf{R}_1$ to $\mathbf{R}_4$ denote the matrices associated with the four formulations. We have

$$\mathbf{R}_1 = \mathbf{M}_1 + \alpha\mathbf{F} + \beta\mathbf{E}, \qquad \mathbf{R}_2 = \mathbf{M}_2 + \alpha\mathbf{I} + \beta\mathbf{J}, \tag{2.8}$$

$$\mathbf{R}_3 = \mathbf{M}_3 + \alpha\mathbf{J} + \beta\mathbf{I}, \qquad \mathbf{R}_4 = \mathbf{M}_4 + \alpha\mathbf{E} + \beta\mathbf{F}. \tag{2.9}$$

2.1. Spectral properties. Let $\sigma(M)$ denote the spectrum of M, $\overline{\sigma(M)}$ the set of complex conjugates of the elements of $\sigma(M)$ and $\overline{M}$ the componentwise complex conjugate of M. The matrix $M\overline{M}$ is important in our analysis. Its spectrum, $\sigma(\overline{M}M)$, is symmetric with respect to the real axis, i.e., if $\mu \in \sigma(\overline{M}M)$, then $\bar{\mu} \in \sigma(\overline{M}M)$. Moreover, the eigenvalues μ and $\bar{\mu}$ are of the same multiplicity. The matrix $\overline{M}M$ is similar to $M\overline{M}$. We refer the reader to [8, Section 4.6] for the proofs of these properties.

LEMMA 2.1. *Let* $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ *and* $\mathbf{J}$ *be the matrices in* (2.5) *and* (2.6). *Assume* $\sigma(M\overline{M}) = \{\mu_i, i = 1, \ldots, m\}$. *Then we have the following facts:*

(1) $\sigma(\mathbf{M}_1) = \sigma(M) \cup \overline{\sigma(M)}$, $\sigma(\mathbf{M}_4) = -\mathrm{i}\sigma(M) \cup \mathrm{i}\overline{\sigma(M)}$.

(2) *If* $\lambda \in \sigma(\mathbf{M}_2)$*, then* $-\lambda, \bar{\lambda}, -\bar{\lambda} \in \sigma(\mathbf{M}_2)$.
(3) *If* $\lambda \in \sigma(\mathbf{M}_3)$*, then* $-\lambda, \bar{\lambda}, -\bar{\lambda} \in \sigma(\mathbf{M}_3)$.
(4) $\sigma(\mathbf{M}_2) = \sigma(\mathbf{J}\mathbf{M}_2) = \sigma(\mathbf{M}_3) = \sigma(\mathbf{J}\mathbf{M}_3) = \{\pm\sqrt{\mu_i}, i = 1, \ldots, m\}$.

Proof. See Proposition 5.1(a)(b) of [7] and Section 1.1 of [4]. □

LEMMA 2.2. *Let* $\omega \in \mathbb{C}$*,* $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4, \mathbf{E}, \mathbf{J}$ *be the matrices in* (2.5) *and* (2.6)*. Assume* $\sigma(M\overline{M}) = \{\mu_i, i = 1, \ldots, m\}$*. Then we have the following facts:*
(1) $\sigma(\mathbf{M}_1 + \omega\mathbf{E}) = \sigma(\mathbf{M}_1 - \omega\mathbf{E})$, $\sigma(\mathbf{M}_4 + \omega\mathbf{E}) = \sigma(\mathbf{M}_4 - \omega\mathbf{E})$.
(2) $\sigma(\mathbf{M}_2 + \omega\mathbf{J}) = \sigma(\mathbf{M}_2 - \omega\mathbf{J}) = \sigma(\mathbf{M}_3 + \omega\mathbf{J}) = \sigma(\mathbf{M}_3 - \omega\mathbf{J}) = \{\pm\sqrt{\mu_i - \omega^2}, i = 1, \ldots, m\}$.

Proof. By $-\mathbf{J}\mathbf{J} = \mathbf{I}$, $-\mathbf{J}(\mathbf{M}_1 + \omega\mathbf{E})\mathbf{J} = \mathbf{M}_1 - \omega\mathbf{E}$, and $-\mathbf{J}(\mathbf{M}_4 + \omega\mathbf{E})\mathbf{J} = \mathbf{M}_4 - \omega\mathbf{E}$, (1) is obvious. Let

$$\mathbf{S} = \frac{\sqrt{2}}{2}\begin{bmatrix} I & -\mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix}. \tag{2.10}$$

By a straightforward calculation, we have

$$\mathbf{S}(\mathbf{M}_2 + \omega\mathbf{J})^2\mathbf{S}^* = \begin{bmatrix} \overline{M}M - \omega^2 I & 0 \\ 0 & M\overline{M} - \omega^2 I \end{bmatrix}, \tag{2.11}$$

and

$$(\mathbf{M}_2 + \omega\mathbf{J})^2 = (\mathbf{M}_2 - \omega\mathbf{J})^2 = \mathbf{M}_2^2 - \omega^2\mathbf{I}. \tag{2.12}$$

By (2.11), (2.12), Lemma 2.1 (4) and $\sigma(M\overline{M}) = \sigma(\overline{M}M)$, we have

$$\sigma(\mathbf{M}_2 + \omega\mathbf{J}) = \sigma(\mathbf{M}_2 - \omega\mathbf{J}) = \{\pm\sqrt{\mu_i - \omega^2}, i = 1, \ldots, m\}.$$

Similarly,

$$\sigma(\mathbf{M}_3 + \omega\mathbf{J}) = \sigma(\mathbf{M}_3 - \omega\mathbf{J}) = \{\pm\sqrt{\mu_i - \omega^2}, i = 1, \ldots, m\}.$$

We obtain (2). □

THEOREM 2.3. *Let* $\alpha, \beta \in \mathbb{R}$*,* $\mathbf{R}_2, \mathbf{R}_3$ *be the matrices in* (2.8) *and* (2.9)*. Assume* $\sigma(M\overline{M}) = \{\mu_i, i = 1, \ldots, m\}$*. Then we have the following facts:*
(1) $\sigma(\mathbf{R}_2) = \{\alpha \pm \sqrt{\mu_i - \beta^2}, i = 1, \ldots, m\}$ *is symmetric with respect to the point* $(\alpha, 0)$ *and the real axis.*
(2) $\sigma(\mathbf{R}_3) = \{\beta \pm \sqrt{\mu_i - \alpha^2}, i = 1, \ldots, m\}$ *is symmetric with respect to the point* $(\beta, 0)$ *and the real axis.*

Proof. (1) and (2) are direct results of Lemma 2.2 (2) and the eigenvalues come in conjugate pairs for real matrices. □

3. $\mathbb{R}$-linear GMRES. In this section, we investigate the $\mathbb{R}$-linear GMRES for solving (1.1) through the R2 formulation (2.2). Let

$$\mathcal{K}_i(\mathcal{M}_\kappa, b) := \mathrm{span}\{b, \mathcal{M}_\kappa b, \ldots, \mathcal{M}_\kappa^{i-1}b\} \subset \mathbb{C}^n,$$

denote the ith Krylov subspace generated by $\mathcal{M}_\kappa$ and $b \in \mathbb{C}^n$. Define $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ by

$$\phi(z) = \begin{bmatrix} \mathrm{Re}(z) \\ \mathrm{Im}(z) \end{bmatrix}.$$

Let

$$\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) := \mathrm{span}^{\mathbb{R}}\{\phi(b), \mathbf{R}_2\phi(b), \dots, \mathbf{R}_2^{i-1}\phi(b)\} \subset \mathbb{R}^{2n},$$

denote the ith Krylov subspace generated by $\mathbf{R}_2$ and $\phi(b) \in \mathbb{R}^{2n}$. Here and in the sequel, the notation "$\mathrm{span}^{\mathbb{R}}\{\cdots\}$" denotes the space of all real linear combinations of the vectors in braces. We list some important properties of the Krylov subspaces $\mathcal{K}_i(\mathcal{M}_\kappa, b)$ and $\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$ in the following lemma. The proof is easy, and is therefore omitted here.

LEMMA 3.1. *Let $\mathcal{M}_\kappa = \kappa I + M\tau$, $b \in \mathbb{C}^n$, $\mathbf{R}_2$ and $\mathbf{J}$ be the matrices in* (2.8) *and* (2.6)*. Then we have the following facts:*

(1) $\mathcal{K}_i(\mathcal{M}_\kappa, b) = \mathrm{span}\{b, M\tau b, \dots, (M\tau)^{i-1}b\}$;
(2) $\mathcal{K}_i(\mathcal{M}_\kappa, b) \subseteq \mathcal{K}_{i+1}(\mathcal{M}_\kappa, b)$;
(3) $\mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b)) \subseteq \mathcal{K}_{i+1}(\mathcal{M}_\kappa, b)$;
(4) $\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \subset \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b)), \mathbf{J}\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \subset \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))$;
(5) $\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) + \mathbf{J}\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$;
(6) $\mathrm{span}\{b\} + \mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \mathcal{K}_{i+1}(\mathcal{M}_\kappa, b)$;
(7) $\mathrm{span}^{\mathbb{R}}\{\phi(b), \mathbf{J}\phi(b)\} + \mathbf{R}_2\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \phi(\mathcal{K}_{i+1}(\mathcal{M}_\kappa, b))$;
(8) *There is a positive integer m such that*

$$\dim(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \begin{cases} i, & 1 \le i \le m, \\ m, & i \ge m+1, \end{cases}$$

$$\dim(\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))) = \begin{cases} 2i, & 1 \le i \le m, \\ 2m, & i \ge m+1, \end{cases}$$

*where "*dim*" denotes dimension.*

We remark that $\mathbf{R}_2\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ is a linear subspace of $\mathbb{R}^{2n}$ and $\mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ is an $\mathbb{R}$-linear subspace of $\mathbb{C}^n$. We can generate the Arnoldi basis of $\mathcal{K}_i(\mathcal{M}_\kappa, b)$ by the Arnoldi process [16, 21].

Let z_0 be the initial guess and $r_0 = b - \mathcal{M}_\kappa z_0$ the corresponding residual. The ith iterate, z_i, determined by RL-GMRES satisfies

$$\|b - \mathcal{M}_\kappa z_i\| = \min_{w \in z_0 + \mathcal{K}_i(\mathcal{M}_\kappa, r_0)} \|b - \mathcal{M}_\kappa w\|, \qquad z_i \in z_0 + \mathcal{K}_i(\mathcal{M}_\kappa, r_0).$$

Let $\widetilde{I}_i$ denote the $i \times i$ identity matrix augmented with the row of zeros as the last row and e_1 the first column of the identity matrix with appropriate dimension. Let $H_{i+1,i}$ be the upper Hessenberg matrix generated in the Arnoldi process (see step 2 of Algorithm 1 below). The ith iterate z_i satisfies

$$\|b - \mathcal{M}_\kappa z_i\| = \min_{s \in \mathbb{C}^i} \left\| \|r_0\| e_1 - \kappa \widetilde{I}_i s - H_{i+1,i}\bar{s} \right\|.$$

The minimal problem

$$\min_{s \in \mathbb{C}^i} \left\| \|r_0\| e_1 - \kappa \widetilde{I}_i s - \widetilde{H}_{i+1,i}\bar{s} \right\|$$

can be solved by employing the $\mathbb{R}$-linear QR decomposition [6]. The work and storage of RL-GMRES (as a function of the number of iterations) are comparable to those of GMRES. We give the details of RL-GMRES in Algorithm 1.

Algorithm 1: **$\mathbb{R}$-linear GMRES**

1. Compute $r_0 = b - \mathcal{M}_\kappa z_0$, z_0 is the initial guess
2. Generate the Arnoldi basis and the matrix $H_{i+1,i}$:

    $v_1 = r_0/\|r_0\|$;
    for $j = 1, 2, \ldots,$ **do**
        $w = M\bar{v}_j$
        for $i = 1$ to j **do**
            $h_{ij} = v_i^* w$
            $w = w - h_{ij} v_i$
        end for
        $h_{j+1,j} = \|w\|$
        $v_{j+1} = w/h_{j+1,j}$
        Solve the minimal problem $\min_{s\in\mathbb{C}^i} \left\| \|r_0\| e_1 - \kappa \widetilde{I}_i s - H_{i+1,i}\bar{s} \right\|$ for s
        Set $z_i = z_0 + V_i s$ and $r_i = b - \mathcal{M}_\kappa z_i$
        Exit if satisfied
    end for

For notational simplicity, in the sequel, we choose the initial guess to be $z_0 = 0$. In step 2 of Algorithm 1, application of i steps of the Arnoldi process with starting vector b yields the Arnoldi decomposition

$$\mathcal{M}_\kappa V_i = V_{i+1}(\kappa \widetilde{I}_i + H_{i+1,i}), \tag{3.1}$$

where $V_i = [v_1, v_2, \ldots, v_i]$ and $V_i^* V_i = I$. The columns of V_i form an orthonormal basis of the Krylov subspace $\mathcal{K}_i(\mathcal{M}_\kappa, b)$; see [6, Theorem 3.1].

We say that the Arnoldi process (3.1) breaks down at step m if $h_{m+1,m} = 0$. Similar to that of GMRES, the exact solution is determined by RL-GMRES when breakdown occurs. However, the proof is not trivial because in general $\mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ is not a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$. When $h_{m+1,m} = 0$,

$$\mathcal{M}_\kappa V_m = V_m(\kappa I + H_m),$$

where H_m is the leading $m \times m$ submatrix of $H_{m+1,m}$. We have

$$\dim(\mathcal{K}_{m+1}(\mathcal{M}_\kappa, b)) = \dim(\mathcal{K}_m(\mathcal{M}_\kappa, b)) = m,$$

and

$$\dim(\phi(\mathcal{K}_{m+1}(\mathcal{M}_\kappa, b))) = \dim(\phi(\mathcal{K}_m(\mathcal{M}_\kappa, b))) = 2m. \tag{3.2}$$

If $\mathbf{R}_2$ is nonsingular, then it follows from $\dim(\mathbf{R}_2\phi(\mathcal{K}_m(\mathcal{M}_\kappa, b))) = 2m$, (3.2) and (7) of Lemma 3.1 that

$$\operatorname{span}^{\mathbb{R}}\{\phi(b), \mathbf{J}\phi(b)\} \subseteq \mathbf{R}_2\phi(\mathcal{K}_m(\mathcal{M}_\kappa, b)).$$

Therefore,

$$\operatorname{span}\{b\} \subseteq \mathcal{M}_\kappa(\mathcal{K}_m(\mathcal{M}_\kappa, b)). \tag{3.3}$$

It follows from (3.3) that

$$\|r_m\| = \|b - \mathcal{M}_\kappa z_m\| = 0.$$

This is referred to as a *benign* breakdown.

In the sequel, $\mathbb{P}_i$ denotes the set of all complex polynomials of degree at most i, and $\mathbb{P}_i^{\mathbb{R}}$ denotes the subset of $\mathbb{P}_i$ of polynomials with real coefficients. The RL-GMRES process chooses the coefficients of $p_{i-1} \in \mathbb{P}_{i-1}$ to minimize the norm of the residual $r_i = b - \mathcal{M}_\kappa(p_{i-1}(\mathcal{M}_\kappa)b)$. The proof of the following proposition is obvious.

Proposition 3.2. *Let* $\mathbf{J}$ *and* $\mathbf{R}_2$ *be the matrices in* (2.6) *and* (2.8), *and* r_i *be the* ith *residual of* RL-GMRES *applied to* (1.1). *Then*

$$\|r_i\| = \min_{w \in \mathcal{K}_i(\mathcal{M}_\kappa, b)} \|b - \mathcal{M}_\kappa w\| = \min_{\phi(w) \in \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))} \|\phi(b) - \mathbf{R}_2\phi(w)\| \tag{3.4}$$

$$= \min_{p_1, p_2 \in \mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b) - \mathbf{R}_2 p_1(\mathbf{R}_2)\phi(b) - \mathbf{R}_2\mathbf{J}p_2(\mathbf{R}_2)\phi(b)\|. \tag{3.5}$$

Let $r_i^{\mathbb{R}}$ be the ith residual of GMRES applied to (2.2), we have

$$\|r_i^{\mathbb{R}}\| = \min_{u \in \mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))} \|\phi(b) - \mathbf{R}_2 u\|. \tag{3.6}$$

By (3.4) and (3.6), RL-GMRES can be viewed as an augmented GMRES method. At step i, the subspace $\mathbf{J}\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$ is added. It follows from $\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \subseteq \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ that

$$\|r_i\| \le \|r_i^{\mathbb{R}}\|. \tag{3.7}$$

This inequality implies that RL-GMRES applied to (1.1) is better than GMRES applied to (2.2). See also Proposition 3.6 of [6]. A sufficient condition for GMRES applied to (2.2) having the same performance as RL-GMRES applied to (1.1) is given in Proposition 3.3. The proof of this proposition is obvious.

Proposition 3.3. *If* $\mathcal{K}_{i+1}^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \perp \mathbf{R}_2\mathbf{J}\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$ *then* $\|r_i\| = \|r_i^{\mathbb{R}}\|$.

Similar to the GMRES approximation problem [20, p.269], we can define the RL-GMRES approximation problem

$$\text{Find } p_{i-1} \in \mathbb{P}_{i-1} \text{ such that } \|b - \mathcal{M}_\kappa(p_{i-1}(\mathcal{M}_\kappa)b)\| = \text{minimum}. \tag{3.8}$$

Theorem 3.4.

(1) RL-GMRES *is scale-invariant, i.e., if* $\mathcal{M}_\kappa$ *is changed to* $\omega\mathcal{M}_\kappa$ *for* $\omega \neq 0 \in \mathbb{C}$, *and* b *is changed to* ωb, *the residuals* $\{r_i\}$ *change to* $\{\omega r_i\}$.

(2) RL-GMRES *is invariant under unitary similarity transformations, i.e., if* $\mathcal{M}_\kappa$ *is changed to* $U\mathcal{M}_\kappa U^*$ *for some unitary matrix* U, *and* b *is changed to* Ub, *the residuals* $\{r_i\}$ *change to* $\{Ur_i\}$.

Proof. By (1) of Lemma 3.1, for any $p_{i-1} \in \mathbb{P}_{i-1}$, there exist $\tilde{p}_{i-1}, \hat{p}_{i-1}, q_{i-1} \in \mathbb{P}_{i-1}$ such that $p_{i-1}(\omega\mathcal{M}_\kappa)(\omega b) = \tilde{p}_{i-1}(\omega M\tau)(\omega b) = \hat{p}_{i-1}(M\tau)b = q_{i-1}(\mathcal{M}_\kappa)b$. Then (1) is readily proved from (3.8). The proof of (2) is easy. □

In view of Theorem 3.4 (1) and Proposition 3.2, if one matrix of the set

$$\{\widetilde{\mathbf{M}}_2 + \tilde{\alpha}\mathbf{I} + \tilde{\beta}\mathbf{J} : \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}, \tilde{\alpha}^2 + \tilde{\beta}^2 = |\kappa|^2\}$$

has desirable properties for GMRES convergence rate, then RL-GMRES applied to (1.1) has speedy convergence rate (see Example 2 of section 4). Here $\tilde{\alpha} + \mathrm{i}\tilde{\beta} = e^{\mathrm{i}\vartheta}(\alpha + \mathrm{i}\beta)$, $\vartheta \in [0, 2\pi)$, $\widetilde{\mathbf{M}}_2 = \Theta\mathbf{M}_2$, and $\Theta = \cos\vartheta\mathbf{I} + \sin\vartheta\mathbf{J}$.

Let $\mathcal{N} = P + Q\tau$, where $P, Q \in \mathbb{C}^{n\times n}$. We have

$$\mathcal{M}_\kappa\mathcal{N} = (\kappa I + M\tau)(P + Q\tau) = (\kappa P + M\overline{Q}) + (\kappa Q + M\overline{P})\tau.$$

To use RL-GMRES, the preconditioned $\mathbb{R}$-linear operator $\mathcal{M}_\kappa\mathcal{N}$ should preserve the form $\omega I + N\tau$, i.e., $\kappa P + M\overline{Q} = \omega I$. On the other hand, if $\kappa Q + M\overline{P} = 0$ then the preconditioned $\mathbb{R}$-linear operator $\mathcal{M}_\kappa\mathcal{N} = \kappa P + M\overline{Q}$. Simplest options for choosing P are diagonal matrices. Choosing $P = \bar{\kappa}I$ and $Q = -M$, we have the linear system (1.2) with $z = (\bar{\kappa}I - M\tau)w$.

To compare RL-GMRES applied to (1.1) with GMRES applied to (1.2), we need the following lemma which describes the shift-invariance property [19] of Krylov subspaces.

Lemma 3.5. *For any $A \in \mathbb{R}^{n\times n}$, $\rho \in \mathbb{R}$, $x \in \mathbb{R}^n$,*

$$\mathcal{K}_i^{\mathbb{R}}(A + \rho I, x) = \mathcal{K}_i^{\mathbb{R}}(A, x),$$

i.e., any $u = p_{i-1}(A+\rho I)x$ with $p_{i-1} \in \mathbb{P}_{i-1}^{\mathbb{R}}$ can be expressed as $u = \tilde{p}_{i-1}(A)x$, where $\tilde{p}_{i-1}(\zeta) = p_{i-1}(\zeta + \rho)$.

In the following proposition, the residual norms of RL-GMRES applied to (1.1) and GMRES applied to (1.2) are expressed by the equivalent real formulations.

Proposition 3.6. *Let $\mathbf{M}_2$, $\mathbf{I}$ be the matrices in* (2.5)-(2.6)*, $\kappa = |\kappa|e^{\mathrm{i}\theta}$, $\Theta = \cos\theta\mathbf{I} - \sin\theta\mathbf{J}$, $\mathbf{M} = \Theta\mathbf{M}_2$ and $b_\theta = e^{-\mathrm{i}\theta}b$.*

(1) *Let r_i^{G} be the ith residual of* GMRES *applied to* (1.2)*. Then,*

$$\|r_i^{\mathrm{G}}\| = \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_1(\mathbf{M}^2)\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_2(\mathbf{M}^2)\phi(\mathrm{i}b_\theta)\|.$$

(2) *Let r_i be the ith residual of* RL-GMRES *applied to* (1.1)*. Then,*

$$\|r_i\| = \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b_\theta) - (\mathbf{M} + |\kappa|\mathbf{I})p_1(\mathbf{M})\phi(b_\theta) - (\mathbf{M} + |\kappa|\mathbf{I})p_2(\mathbf{M})\phi(\mathrm{i}b_\theta)\|.$$

Proof. (1) Let $C = |\kappa|^2 I - M\overline{M}$, $\mathbf{S}$ be the matrix in (2.10) and $\mathbf{C} = \begin{bmatrix} \overline{C} & 0 \\ 0 & C \end{bmatrix}$. Then,

$$\begin{aligned}
\|r_i^{\mathrm{G}}\| &= \min_{p_{i-1}\in\mathbb{P}_{i-1}} \|b - Cp_{i-1}(C)b\| = \min_{p_{i-1}\in\mathbb{P}_{i-1}} \|b_\theta - Cp_{i-1}(C)b_\theta\| \\
&= \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|b_\theta - Cp_1(C)b_\theta - \mathrm{i}Cp_2(C)b_\theta\| \\
&= \frac{\sqrt{2}}{2} \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \left\| \begin{bmatrix} \overline{b_\theta} - \overline{C}p_1(\overline{C})\overline{b_\theta} + \mathrm{i}\overline{C}p_2(\overline{C})\overline{b_\theta} \\ b_\theta - Cp_1(C)b_\theta - \mathrm{i}Cp_2(C)b_\theta \end{bmatrix} \right\| \\
&= \frac{\sqrt{2}}{2} \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \left\| \begin{bmatrix} \overline{b_\theta} - \overline{C}p_1(\overline{C})\overline{b_\theta} + \mathrm{i}\overline{C}p_2(\overline{C})\overline{b_\theta} \\ -\mathrm{i}b_\theta - Cp_1(C)(-\mathrm{i}b_\theta) - \mathrm{i}Cp_2(C)(-\mathrm{i}b_\theta) \end{bmatrix} \right\| \\
&= \frac{\sqrt{2}}{2} \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \left\| \begin{bmatrix} \overline{b_\theta} \\ -\mathrm{i}b_\theta \end{bmatrix} - \begin{bmatrix} \overline{C}p_1(\overline{C})\overline{b_\theta} \\ Cp_1(C)(-\mathrm{i}b_\theta) \end{bmatrix} + \begin{bmatrix} \overline{C}p_2(\overline{C})\mathrm{i}\overline{b_\theta} \\ Cp_2(C)(-b_\theta) \end{bmatrix} \right\| \\
&= \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\mathbf{S}\phi(b_\theta) - \mathbf{C}p_1(\mathbf{C})\mathbf{S}\phi(b_\theta) - \mathbf{C}p_2(\mathbf{C})\mathbf{S}\phi(\mathrm{i}b_\theta)\| \\
&= \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_1(\mathbf{M}^2)\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_2(\mathbf{M}^2)\phi(\mathrm{i}b_\theta)\|.
\end{aligned}$$

The last equality holds due to (2.11)-(2.12), Lemma 3.5 and $\mathbf{M}_2^2 = \mathbf{M}^2$.

(2) Note that $\mathbf{M} + |\kappa|\mathbf{I}$ is the coefficient matrix of the R2 formulation of the following $\mathbb{R}$-linear system

$$|\kappa|z + e^{-\mathrm{i}\theta}M\bar{z} = e^{-\mathrm{i}\theta}b.$$

Then the result follows from Theorem 3.4 (1), Proposition 3.2, Lemma 3.5, $\mathbf{JM} = -\mathbf{MJ}$ and $\mathbf{J}\phi(b_\theta) = \phi(\mathrm{i}b_\theta)$. ∎

For the case of $\kappa = 0$, (1.1) and (1.2) reduce to

$$M\bar{z} = b, \tag{3.9}$$

and

$$-M\overline{M}w = b, \tag{3.10}$$

respectively. By Proposition 3.6, we immediately obtain the following corollary.

Corollary 3.7. *Let* r_i *and* r_i^{G} *be the* ith *residual of* RL-GMRES *applied to* (3.9) *and the* ith *residual of* GMRES *applied to* (3.10)*, respectively. Then,* $\|r_{2i}\| \le \|r_i^{\mathrm{G}}\|$.

Actually, the inequality $\|r_{2i}\| \le \|r_i^{\mathrm{G}}\|$ holds for any $\kappa \in \mathbb{C}$. To prove this, we need the following lemma.

Lemma 3.8. *Let* $\mathbf{M}$ *and* $\mathbf{I}$ *be the matrices in* Proposition 3.6*. Then, for any* $p \in \mathbb{P}^{\mathbb{R}}_{i-1}$*, there exists a polynomial* $\tilde{p} \in \mathbb{P}^{\mathbb{R}}_{2i-1}$ *satisfying*

$$(\mathbf{M} + |\kappa|\mathbf{I})\tilde{p}(\mathbf{M}) = (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p(\mathbf{M}^2).$$

Proof. Note that for any $p \in \mathbb{P}^{\mathbb{R}}_{i-1}$, there exists a polynomial $\tilde{p} \in \mathbb{P}^{\mathbb{R}}_{2i-1}$ such that

$$\tilde{p}(\mathbf{M}) = (\mathbf{M} - |\kappa|\mathbf{I})p(\mathbf{M}^2). \tag{3.11}$$

Multiplying both sides of (3.11) by $\mathbf{M} + |\kappa|\mathbf{I}$, we complete the proof. ∎

Theorem 3.9. *Let* r_i *and* r_i^{G} *be the* ith *residual of* RL-GMRES *applied to* (1.1) *and the* ith *residual of* GMRES *applied to* (1.2)*, respectively. Then,*

$$\|r_{2i}\| \le \|r_i^{\mathrm{G}}\|.$$

Proof. This is a direct result of Proposition 3.6 and Lemma 3.8. ∎

Remark 3.10. By Theorem 3.9, RL-GMRES applied to (1.1) requires fewer matrix-vector products than GMRES applied to (1.2) (Note that GMRES applied to (1.2) requires two matrix-vector products every iteration, and for z an extra matrix-vector product is required). See the numerical experiments in section 4.

Remark 3.11. We can prove Theorem 3.9 through a different approach by the shift-invariance property of Krylov subspaces and the minimal residual property of RL-GMRES and GMRES; see [5, 15].

4. Numerical experiments. In this section, we report numerical results of two examples. Throughout, the computation is performed in MATLAB 2008a on a laptop with 2.26G CPU and 4GB memory.

4.1. Example 1. The matrix M used in this example is a randomly chosen complex tridiagonal matrix of dimension $n = 200$. The right-hand side b is a randomly chosen complex vector. The matrix M and the right-hand side b are generated by the following MATLAB codes.

```
n = 200 ;
d1 = rand(n,1)+i*rand(n,1);
d2 = rand(n-1,1)+i*rand(n-1,1);
d3 = rand(n-1,1)+i*rand(n-1,1);
M = diag(d1)+diag(d2,-1)+diag(d3,1);
b = rand(n,1)+i*rand(n,1);
```

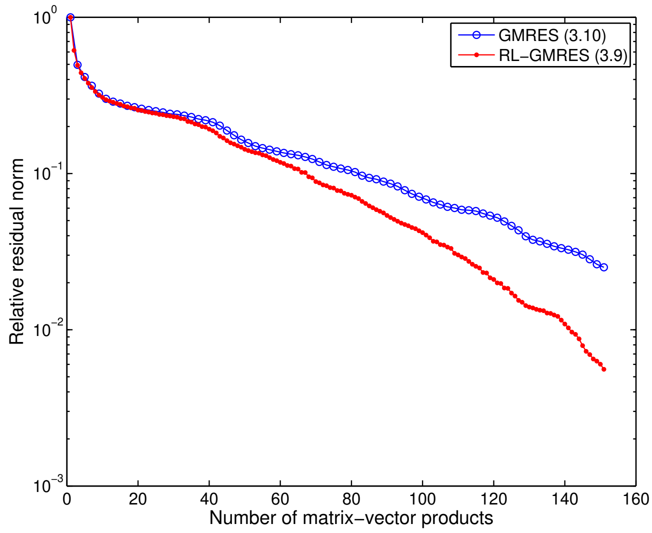

FIG. 4.1. *Example* 1. *Convergence history for* GMRES *applied to* (3.10) *and* RL-GMRES *applied to* (3.9).

We compare RL-GMRES applied to (3.9) with GMRES applied to (3.10). The initial guess is set to be the zero vector and the iteration stops if the number of matrix-vector products is 150. Note that GMRES applied to (3.10) requires two matrix-vector products every iteration and RL-GMRES requires only one. We have tested this example many times and Figure 4.1 shows the typical convergence history of these methods. The curve of RL-GMRES applied to (3.9) is below the curve of GMRES applied to (3.10), which illustrates Corollary 3.7.

4.2. Example 2. In this example, we consider an ℝ-linear system arising from the inverse problem of reconstructing an unknown electric conductivity [1, 11]. More precisely, we need to solve the ℝ-linear integral equation

$$u + (\bar{\nu}_1 \mathcal{I}_1 + \bar{\nu}_2 \mathcal{I}_2)\bar{u} = \bar{\nu}_3, \tag{4.1}$$

defined in the rectangle $\Omega = [-1, 1)^2$. The integral operators $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined as follows, for $f \in C_0^\infty(\mathbb{C})$,

$$\mathcal{I}_1 f(\xi) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\zeta-\xi|>\varepsilon} \frac{f(\zeta)}{(\zeta-\xi)^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2, \quad \mathcal{I}_2 f(\xi) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - \xi} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2,$$

where $\zeta = \zeta_1 + \mathrm{i}\zeta_2$. We remark that $\mathcal{I}_1$ (Beurling transform) is a singular integral operator and $\mathcal{I}_2$ (Cauchy transform) is a weakly singular integral operator.

The numerical discretization of (4.1) on uniform $n \times n$ grids results in ℝ-linear systems of the form

$$z + (D_1 T_1 + D_2 T_2)\bar{z} = b, \tag{4.2}$$

where D_1 and D_2 are diagonal matrices, and T_1 and T_2 are block-Toeplitz-Toeplitz-block (BTTB) matrices. More precisely, T_1 is complex symmetric and has the structure

$$T_1 = -\frac{1}{\pi}\begin{bmatrix} M_1 & M_2 & \cdots & M_{n-1} & M_n \\ M_2^T & M_1 & M_2 & \cdots & M_{n-1} \\ \vdots & M_2^T & M_1 & \ddots & \vdots \\ M_{n-1}^T & \cdots & \ddots & \ddots & M_2 \\ M_n^T & M_{n-1}^T & \cdots & M_2^T & M_1 \end{bmatrix},$$

where $M_k = (m_{ij}^k)$, $i, j, k = 1, \ldots, n$ are Toeplitz matrices with

$$m_{ij}^k = \begin{cases} \dfrac{1}{(j-i+(k-1)\mathrm{i})^2}, & k \neq 1 \text{ or } i \neq j, \\ 0, & k = 1 \text{ and } i = j. \end{cases}$$

The matrix T_2 is complex skew-symmetric and has the structure

$$T_2 = -\frac{h}{\pi}\begin{bmatrix} N_1 & N_2 & \cdots & N_{n-1} & N_n \\ -N_2^T & N_1 & N_2 & \cdots & N_{n-1} \\ \vdots & -N_2^T & N_1 & \ddots & \vdots \\ -N_{n-1}^T & \cdots & \ddots & \ddots & N_2 \\ -N_n^T & -N_{n-1}^T & \cdots & -N_2^T & N_1 \end{bmatrix},$$

where $h = 2/n$, $N_k = (n_{ij}^k)$, $i, j, k = 1, \ldots, n$ are Toeplitz matrices with

$$n_{ij}^k = \begin{cases} \dfrac{1}{j-i+(k-1)\mathrm{i}}, & k \neq 1 \text{ or } i \neq j, \\ 0, & k = 1 \text{ and } i = j. \end{cases}$$

Let $\nu_1(\zeta) = -\exp(-\mathrm{i}(k\zeta + \bar{k}\bar{\zeta}))\dfrac{1-\varphi(\zeta)}{1+\varphi(\zeta)}$, $\nu_2(\zeta) = -\mathrm{i}\bar{k}\nu_1(\zeta)$, $\nu_3(\zeta) = -\nu_2(\zeta)$, where $k \in \mathbb{C}$ is a parameter, and the piecewise continuous conductivity $\varphi(\zeta)$ in the unit square is

$$\varphi(\zeta) = \begin{cases} 3, & |\zeta + 0.3\mathrm{i}| < 0.3, \\ 0.3, & |\zeta + 0.4 - 0.3\mathrm{i}| < 0.3 \text{ or } |\zeta - 0.4 - 0.3\mathrm{i}| < 0.3, \\ 1, & \text{otherwise}. \end{cases}$$

We have $M = D_1 T = D_1(T_1 + \mathrm{i}kT_2)$. The multiplication by M consists of a multiplication by the BTTB matrix T followed by a diagonal matrix D_1, which can be obtained in $\mathcal{O}(n^2 \log n)$ operations by FFT. We compare the performance of RL-GMRES applied to (1.1) with GMRES applied to (1.2). The initial guess is set to be the zero vector and the iteration stops if $\|b - z_i - M\bar{z}_i\|/\|b\| \leq 10^{-12}$ (RL-GMRES) or $\|b - |\kappa|^2 w_i + M\overline{M}w_i\|/\|b\| \leq 10^{-12}$ (GMRES).

We plot in Figure 4.2 the eigenvalues of the matrices $I - M\overline{M}$ and $\mathbf{R}_2$ arising from the numerical discretization on a 64×64 grid. The spectral radius of $M\overline{M}$, $\rho(M\overline{M}) \approx 0.3509 < 1$. It is obvious that $\sigma(I - M\overline{M})$ is located in the right-half plane. By Theorem 2.3, $\sigma(\mathbf{R}_2)$ is located in the right-half plane. See Figure 4.2. Table 4.1 shows the numbers of matrix-vector products of RL-GMRES applied to (1.1) and GMRES applied to (1.2) for numerical discretizations of (4.1) on uniform $N \times N$ grids. We observe that both RL-GMRES and GMRES converge fast and RL-GMRES is slightly faster than GMRES in terms of matrix-vector products.

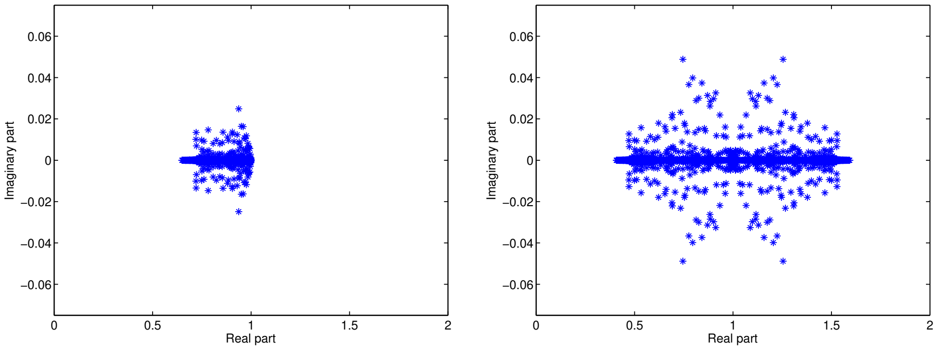

FIG. 4.2. *Example* 2. *Eigenvalues of* $I - M\overline{M}$ (*left*) *and* $\mathbf{R}_2$ (*right*)

TABLE 4.1
Example 2. *Number of matrix-vector products of* RL-GMRES *applied to* (1.1) *and* GMRES *applied to* (1.2) *for numerical discretizations of* (4.1) *on uniform* $n \times n$ *grids.*

Method	64×64	128×128	256×256	512×512
RL-GMRES	26	24	24	23
GMRES	27	27	25	25

5. Conclusions. We have analyzed $\mathbb{R}$-linear GMRES for (1.1) through the equivalent real formulation (2.2). We have proved that RL-GMRES applied to the $\mathbb{R}$-linear system (1.1) is faster than GMRES applied to the related $\mathbb{C}$-linear system (1.2) in terms of matrix-vector products. For many challenging $\mathbb{R}$-linear systems such that $\|M\| \gg |\kappa|$ (see [1]), efficient preconditioning techniques are being considered.

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REFERENCES

[1] K. Astala, J. Mueller, L. Päivärinta, and S. Siltanen, *Numerical computation of complex geometrical optics solutions to the conductivity equation*, Appl. Comp. Harm. Anal., to appear (2010).
[2] O. Axelsson and A. Kucherov, *Real valued iterative methods for solving complex symmetric linear systems*, Numer. Linear Algebra Appl., 7 (2000), pp. 197–218.
[3] M. Benzi and D. Bertaccini, *Block preconditioning of real-valued iterative algorithms for complex linear systems*, IMA J. Numer. Anal., 28 (2008), pp. 598–618.
[4] D. Day and M. A. Heroux, *Solving complex-valued linear systems via equivalent real formulations*, SIAM J. Sci. Comput., 23 (2001), pp. 480–498.
[5] K. Du, *Global $\mathbb{R}$-linear GMRES for solving a class of $\mathbb{R}$-linear matrix equations*, in preparation, (2010).
[6] T. Eirola, M. Huhtanen, and J. von Pfaler, *Solution methods for $\mathbb{R}$-linear problems in $\mathbb{C}^n$*, SIAM J. Matrix Anal. Appl., 25 (2004), pp. 804–828.
[7] R. W. Freund, *Conjugate gradient-type methods for linear systems with complex symmetric coefficient matrices*, SIAM J. Sci. Statist. Comput., 13 (1992), pp. 425–448.
[8] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, 1990.
[9] M. Huhtanen and O. Nevanlinna, *Real linear matrix analysis*, Banach Center Publ., 75 (2007), pp. 171–189.
[10] M. Huhtanen and O. Nevanlinna, *The real linear resolvent and cosolvent operators*, J. Operator Theory, 58 (2007), pp. 229–250.

[11] M. Huhtanen and A. Perämäki, *Numerical solution of the $\mathbb{R}$-linear Beltrami equation*, submitted, (2010).
[12] M. Huhtanen and S. Ruotsalainen, *Real linear operator theory and its applications*, submitted, (2010).
[13] K. Knudsen, J. Mueller, and S. Siltanen, *Numerical solution method for the dbar-equation in the plane*, J. Comput. Phys., 198 (2004), pp. 500–517.
[14] J. Mueller and S. Siltanen, *Direct reconstructions of conductivities from boundary measurements*, SIAM J. Sci. Comput., 24 (2003), pp. 1232–1266.
[15] A. Perämäki, Institute of Mathematics, Aalto University, Personal Communication, (2010).
[16] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, Philadelphia, second ed., 2003.
[17] Y. Saad and M. H. Schultz, *GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 856–869.
[18] S. Siltanen, J. Mueller, and D. Isaacson, *An implementation of the reconstruction algorithm of A. Nachman for the 2D inverse conductivity problem*, Inverse Problems, 16 (2000), pp. 681–699.
[19] V. Simoncini and D. B. Szyld, *Recent computational developments in Krylov subspace methods for linear systems*, Numer. Linear Algebra Appl., 14 (2007), pp. 1–59.
[20] L. N. Trefethen and D. Bau, *Numerical linear algebra*, SIAM, 1997.
[21] H. A. van der Vorst, *Iterative Krylov Methods for Large Linear Systems*, Cambridge University Press, Cambridge, 2003.

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