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AMS subject classifications: 65N15, 76M12, 76S05

Keywords: Stokes problem, a posteriori error estimate, guaranteed upper bound, unified framework, conforming finite element method, discontinuous Galerkin method, nonconforming finite element method, mixed finite element method, finite volume method

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A unified framework for a posteriori error estimation for the Stokes problem

Abstract In this paper, a unified framework for a posteriori error estimation for the Stokes problem is developed. It is based on $[H_0^1(\Omega)]^d$-conforming velocity reconstruction and $\underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$-conforming, locally conservative flux (stress) reconstruction. It gives guaranteed, fully computable global upper bounds as well as local lower bounds on the energy error. In order to apply this framework to a given numerical method, two simple conditions need to be checked. We show how to do this for various conforming and conforming stabilized finite element methods, the discontinuous Galerkin method, the Crouzeix–Raviart nonconforming finite element method, the mixed finite element method, and a general class of finite volume methods. Numerical experiments illustrate the theoretical developments.

Keywords Stokes problem · a posteriori error estimate · guaranteed upper bound · unified framework · conforming finite element method · discontinuous Galerkin method · nonconforming finite element method · mixed finite element method · finite volume method

Mathematics Subject Classification (2000) 65N15, 76M12, 76S05

1 Introduction

The purpose of this paper is to develop a unified framework for a posteriori error estimation for the Stokes problem discretized by different numerical methods. In particular, we apply this framework to conforming divergence-free, discontinuous Galerkin, conforming (stabilized), nonconforming, mixed, and finite volume methods. Our estimates give a guaranteed (that is, not featuring any undetermined constant) upper bound on the error measured in the energy (semi-)norm. They are easily, fully, and locally computable. They are also locally efficient in the sense that they represent local lower bounds for the energy error. Numerical experiments show that their effectivity index (the ratio of the estimated and exact error) is relatively close to the optimal value of one.

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Our estimates are based on $[H_0^1(\Omega)]^d$-conforming velocity reconstruction and $\underline{\boldsymbol{H}}(\mathrm{div},\Omega)$-conforming, locally conservative flux (stress) reconstructions. Such an approach has recently become popular in the framework of second-order elliptic equations, see, e.g., [37,32,43,4,23,38,2,36,50,3,27,51] and the references cited therein. Its main ideas are very physical and can be traced back at least to the Prager–Synge equality [41]. Equilibrated flux estimates have recently been shown to be robust with respect to inhomogeneities, anisotropies, and reaction or convection dominance in [53,20,28] and with respect to the polynomial degree in [13]. In a unifying spirit, similar to the present paper, they have been extended to the heat equation in [30]. Stokes a posterior error estimates related to the present approach have previously been studied in [25,42,10]. However, these estimates are valid only for certain type of numerical approximations.

Locally conservative $\underline{\boldsymbol{H}}(\mathrm{div},\Omega)$-conforming flux reconstruction is straightforward in so-called locally conservative methods [2,50,36,3,27,52,29,28,54,30]. For finite element-type methods, which are not locally conservative by construction, this is less straightforward. However, for such methods, the reconstruction can be achieved by the equilibration procedure, see [4,23,13] and the references therein. We follow here the approach for lowest-order methods of [38,51,53], where no equilibration is needed. We generalize this approach here to higher-order methods. It turns out that only small local problems of fixed size $(d+1)\times(d+1)$ for each mesh element, where d is the space dimension, need to be solved in order to obtain the equilibrated side normal fluxes.

This paper is organized as follows. In Section 2, we state the considered Stokes problem. In Section 3, we specify our notation and give some preliminary results. Sections 4 and 5 collect our a posteriori error estimates, first for conforming divergence-free approximations and then for arbitrary ones. These results are stated in a general form independent of the numerical method at hand; we only suppose the existence of a locally conservative $\underline{\boldsymbol{H}}(\mathrm{div},\Omega)$-conforming flux reconstruction $\underline{\boldsymbol{\sigma}}_h$ (cf. assumptions (4.3) and (5.10) below). Section 6 then presents the efficiency of the estimates, still in a general form independent of the numerical method at hand, only based on Assumption 6.2. In Section 7, we apply the previous results to different numerical methods. This consists in specifying the way of construction of $\underline{\boldsymbol{\sigma}}_h$ and in verifying the assumptions (4.3) or (5.10) and Assumption 6.2. Section 8 presents numerical experiments. A technical result on the the inf–sup condition is proven in Appendix A.

2 The Stokes problem

Here, we describe the Stokes problem considered in this paper. We use standard notation; some details on the notation are given in Section 3 below.

Let $\Omega\subset\mathrm{R}^d$, $d=2,3$, be a polygonal (polyhedral) domain (open, bounded, and connected set). We consider the Stokes problem: given $\mathbf{f}\in[L^2(\Omega)]^d$, find $\mathbf{u}$, the "velocity", and p, the "pressure", such that

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f}\quad\text{in}\quad\Omega,\tag{2.1a}$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in}\quad\Omega,\tag{2.1b}$$

$$\mathbf{u}=\mathbf{0}\quad\text{on}\quad\partial\Omega.\tag{2.1c}$$

Denote by $\mathbf{V}$ the space $[H_0^1(\Omega)]^d$ and by Q the space of $L^2(\Omega)$ functions having zero mean value over Ω. For $\mathbf{u},\mathbf{v}\in\mathbf{V}$ and $q\in Q$, set

$$a(\mathbf{u},\mathbf{v}):=(\nabla\mathbf{u},\nabla\mathbf{v}),\tag{2.2a}$$

$$b(\mathbf{v},q):=-(q,\nabla\cdot\mathbf{v}).\tag{2.2b}$$

The weak formulation of (2.1a)–(2.1c) reads: find $(\mathbf{u},p)\in\mathbf{V}\times Q$ such that

$$a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)=(\mathbf{f},\mathbf{v})\quad\forall\mathbf{v}\in\mathbf{V},\tag{2.3a}$$

$$b(\mathbf{u},q)=0\qquad\forall q\in Q.\tag{2.3b}$$

The above problem is well-posed (cf. [31]) due to the inf–sup condition

$$\inf_{q\in Q}\sup_{\mathbf{v}\in\mathbf{V}}\frac{b(\mathbf{v},q)}{\|\nabla\mathbf{v}\|\,\|q\|}\geq\beta,\tag{2.4}$$

where β is a positive constant. Denote the divergence-free subspace of $\mathbf{V}$ by

$$\mathbf{V}_0 := \{\mathbf{v} \in \mathbf{V};\, \nabla\cdot\mathbf{v} = 0\}.$$

The velocity $\mathbf{u}$ can be equivalently characterized as: find $\mathbf{u} \in \mathbf{V}_0$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0. \tag{2.5}$$

Recall also that by introducing the "stress" tensor $\underline{\boldsymbol{\sigma}} \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$, the problem (2.1a)–(2.1c) can be written as a system consisting of the constitutive law

$$\underline{\boldsymbol{\sigma}} = \nabla\mathbf{u} - p\underline{\boldsymbol{I}}, \tag{2.6}$$

the equilibrium equation

$$\nabla\cdot\underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}, \tag{2.7}$$

and the divergence constraint

$$\nabla\cdot\mathbf{u} = 0, \tag{2.8}$$

for which the pressure p is the Lagrange multiplier. Here $\underline{\boldsymbol{I}}$ is the $d \times d$ identity matrix. Alternatively, (2.6)–(2.7) may be replaced by

$$\underline{\boldsymbol{\sigma}}' = \nabla\mathbf{u} \tag{2.9}$$

and

$$\nabla\cdot\underline{\boldsymbol{\sigma}}' - \nabla p + \mathbf{f} = \mathbf{0}. \tag{2.10}$$

3 Notation and preliminaries

Here, we summarize the notation used throughout the paper and give some preliminary results.

3.1 Notation

Let $D \subset \mathrm{R}^d$. By $(\cdot,\cdot)_D$, we denote the scalar product in $L^2(D)$: $(p,q)_D := \int_D pq\,\mathrm{d}\mathbf{x}$. When D coincides with Ω, the subscript Ω will be dropped. We use the same symbol $(\cdot,\cdot)_D$ for the scalar product in $\mathbf{L}^2(D) := [L^2(D)]^d$ and in $\underline{\boldsymbol{L}}^2(D) := [L^2(D)]^{d\times d}$. More precisely, $(\mathbf{u},\mathbf{v})_D := \sum_{i=1}^d (\mathbf{u}^i, \mathbf{v}^i)_D$ for $\mathbf{u},\mathbf{v} \in \mathbf{L}^2(D)$ and $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}})_D := \sum_{i=1}^d \sum_{j=1}^d (\underline{\boldsymbol{\sigma}}^{i,j}, \underline{\boldsymbol{\tau}}^{i,j})_D$ for $\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{L}}^2(D)$. The associated norm is denoted by $\|\cdot\|_D$. We denote by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(D)$, $D \subset \mathrm{R}^{d-1}$, and its vector and tensor versions. For vectors $\mathbf{u},\mathbf{v} \in \mathrm{R}^d$, $\mathbf{u}\otimes\mathbf{v}$ defines a tensor $\underline{\boldsymbol{\sigma}} \in \mathrm{R}^{d\times d}$ such that $\underline{\boldsymbol{\sigma}}^{i,j} := \mathbf{u}^i\mathbf{v}^j$. Finally, for $D \subset \mathrm{R}^{d'}$, $1 \le d' \le d$, $|D|$ stands for the d'-dimensional Lebesgue measure of D and we denote by $\mathbf{e}_i \in \mathrm{R}^d$ the i-th Euclidean unit vector.

Let $\mathcal{T}_h$ be a polygonal (polyhedral) partition of Ω, whose elements can be nonconvex or non star-shaped. The partition $\mathcal{T}_h$ can be nonmatching, that is, the intersection of two elements T, T' of $\mathcal{T}_h$ is not necessarily their common face, edge, or vertex or an empty set (so-called hanging nodes are allowed). We denote by h_T the diameter of $T \in \mathcal{T}_h$. We say that F is an interior side of $\mathcal{T}_h$ if it has a positive $(d-1)$-dimensional Lebesgue measure and if there are distinct $T^-(F)$ and $T^+(F)$ in $\mathcal{T}_h$ such that $F = \partial T^-(F) \cap \partial T^+(F)$. We define $\mathbf{n}_F$ as the unit normal vector to F pointing from $T^-(F)$ towards $T^+(F)$. Similarly, we say that F is a boundary side of $\mathcal{T}_h$ if it has a positive $(d-1)$-dimensional Lebesgue measure and if there is $T(F) \in \mathcal{T}_h$ such that $F = \partial T(F) \cap \partial\Omega$ and we define $\mathbf{n}_F$ as the unit outward normal to $\partial\Omega$. The arbitrariness in the orientation of $\mathbf{n}_F$ is irrelevant in the sequel. All the interior (resp., boundary) sides of the mesh are collected into the set $\partial\mathcal{T}_h^{\mathrm{int}}$ (resp., $\partial\mathcal{T}_h^{\mathrm{ext}}$) and we set $\partial\mathcal{T}_h := \partial\mathcal{T}_h^{\mathrm{int}} \cup \partial\mathcal{T}_h^{\mathrm{ext}}$. For $F \in \partial\mathcal{T}_h$, h_F stands for its diameter. For $T \in \mathcal{T}_h$, we denote by $\mathcal{F}_T$ all its sides and by $\mathcal{F}_T^{\mathrm{int}}$ those sides of T which belong to $\partial\mathcal{T}_h^{\mathrm{int}}$. We will also use the notation $\mathfrak{T}_T$ (resp., $\mathfrak{F}_T$) for the elements (resp., sides) of $\mathcal{T}_h$ sharing a vertex with T. We denote by $\mathfrak{F}_T^{\mathrm{int}}$ those sides of $\mathfrak{F}_T$ which belong to $\partial\mathcal{T}_h^{\mathrm{int}}$. The notation $\mathcal{V}_h$ (resp., $\mathcal{V}_h^{\mathrm{int}}$) will be used for the set of all (resp., interior) vertices of $\mathcal{T}_h$. Let $V \in \mathcal{V}_h$. Then $\mathfrak{T}_V$ denotes all the elements of $\mathcal{T}_h$ having V as vertex.

For a (sufficiently smooth) scalar, vector, or tensor function v that is double-valued on an interior side F, its jump and average on F are defined as

$$[\![v]\!]_F := v|_{T^-(F)} - v|_{T^+(F)}, \qquad \{\!\{v\}\!\}_F := \tfrac{1}{2}(v|_{T^-(F)} + v|_{T^+(F)}). \tag{3.1}$$

We set $[\![v]\!]_F := v|_F$ and $\{\!\{v\}\!\}_F := v|_F$ on boundary sides. The subscript F in the above jumps and averages is omitted if there is no ambiguity. We denote by $\mathbf{V}(\mathcal{T}_h)$ the space of piecewise smooth vector functions on $\mathcal{T}_h$

$$\mathbf{V}(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega);\ \mathbf{v}_h|_T \in [H^1(T)]^d \quad \forall T \in \mathcal{T}_h\}.$$

Note that $\mathbf{V}(\mathcal{T}_h) \not\subset \mathbf{V}$. We employ the notation $\mathbb{P}_k(\mathcal{T}_h)$ for piecewise polynomials of order k on $\mathcal{T}_h$. In the sequel, we use the signs ∇, Δ, and $\nabla\cdot$ respectively for the elementwise gradient, Laplace, and divergence operators. Some additional notation will also be introduced later where needed.

3.2 Preliminaries

Let $T \in \mathcal{T}_h$ and denote by $\boldsymbol{\varphi}_T$ the average of $\boldsymbol{\varphi}$ over T, i.e., $\boldsymbol{\varphi}_T^i = (\boldsymbol{\varphi}, \mathbf{e}_i)_T/|T|$, $i = 1, \ldots, d$. Then the Poincaré inequality states

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_T\|_T \leq C_{\mathrm{P},T} h_T \|\nabla \boldsymbol{\varphi}\|_T \quad \forall \boldsymbol{\varphi} \in [H^1(T)]^d, \tag{3.2}$$

where the constant $C_{\mathrm{P},T}$ is independent of h_T. It depends only on the shape of T. For a convex T, we have the estimate $C_{\mathrm{P},T} \leq 1/\pi$ [39,9].

Set

$$\mathcal{B}((\mathbf{v},q),(\mathbf{z},r)) := a(\mathbf{v},\mathbf{z}) + b(\mathbf{z},q) + b(\mathbf{v},r). \tag{3.3}$$

The problem (2.3a)–(2.3b) can then be stated as: find $(\mathbf{u},p) \in \mathbf{V} \times Q$ such that

$$\mathcal{B}((\mathbf{u},p),(\mathbf{v},q)) = (\mathbf{f},\mathbf{v}) \quad \forall (\mathbf{v},q) \in \mathbf{V} \times Q. \tag{3.4}$$

We define the energy (semi-)norm for $(\mathbf{v},q) \in \mathbf{V}(\mathcal{T}_h) \times Q$ as

$$|||(\mathbf{v},q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2, \tag{3.5}$$

where β is the constant from the inf–sup condition (2.4). We refer to Appendix A for the proof of the following stability estimate:

Lemma 3.1 (The inf–sup **condition on $\mathbf{V} \times Q$)** *There is a positive constant C_S such that*

$$\inf_{(\mathbf{v},q)\in \mathbf{V}\times Q} \sup_{(\mathbf{z},r)\in \mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{v},q),(\mathbf{z},r))}{|||(\mathbf{z},r)|||\,|||(\mathbf{v},q)|||} \geq C_\mathrm{S} \tag{3.6}$$

with

$$2\sqrt{3} \geq \frac{1}{C_\mathrm{S}}. \tag{3.7}$$

4 A posteriori error estimate for conforming divergence-free approximations

In this section, we derive an a posteriori error estimate valid for arbitrary conforming and divergence-free approximations, i.e., approximation $\mathbf{u}_h \in \mathbf{V}_0$. It can be considered as an intermediate result, as standard approximation methods do not lead to $\mathbf{u}_h \in \mathbf{V}_0$. There exist, however, methods fulfilling this constraint, like that of [44].

Given an approximation $(\mathbf{u}_h, p_h) \in \mathbf{V}_0 \times Q$, not necessarily the numerical solution, the a posteriori error estimators on $T \in \mathcal{T}_h$ are defined as follows. Let $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$. We define the *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T, \tag{4.1}$$

where $C_{\mathrm{P},T}$ is the constant from the the Poincaré inequality (3.2), and the *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T. \tag{4.2}$$

We then have the following estimate.

Theorem 4.1 (Velocity estimate for conforming divergence-free approximations.) *Let $\mathbf{u} \in \mathbf{V}_0$ be the weak solution given by (2.5) and let $(\mathbf{u}_h, p_h) \in \mathbf{V}_0 \times Q$ be arbitrary. Let $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$ be such that*

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h. \tag{4.3}$$

Then

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\| \leq \left\{ \sum_{T \in \mathcal{T}_h} (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 \right\}^{1/2}. \tag{4.4}$$

Proof Using (2.5), we have

$$\begin{aligned}
\|\nabla(\mathbf{u} - \mathbf{u}_h)\| &= a\left(\mathbf{u} - \mathbf{u}_h, \frac{\mathbf{u} - \mathbf{u}_h}{\|\nabla(\mathbf{u} - \mathbf{u}_h)\|} \right) \\
&\leq \sup_{\boldsymbol{\varphi} \in \mathbf{V}_0} \frac{a(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\varphi})}{\|\nabla \boldsymbol{\varphi}\|} \\
&= \sup_{\boldsymbol{\varphi} \in \mathbf{V}_0} \frac{(\mathbf{f}, \boldsymbol{\varphi}) - a(\mathbf{u}_h, \boldsymbol{\varphi})}{\|\nabla \boldsymbol{\varphi}\|}.
\end{aligned}$$

Let $\boldsymbol{\varphi} \in \mathbf{V}_0$ be fixed. Then, using that $\nabla \cdot \boldsymbol{\varphi} = 0$,

$$0 = (p_h, \nabla \cdot \boldsymbol{\varphi}) = (p_h \underline{\boldsymbol{I}}, \nabla \boldsymbol{\varphi}).$$

Moreover, using the Green theorem $(\underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{\varphi}) = -(\nabla \cdot \underline{\boldsymbol{\sigma}}_h, \boldsymbol{\varphi})$ and adding and subtracting $(\underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{\varphi})$,

$$\begin{aligned}
&(\mathbf{f}, \boldsymbol{\varphi}) - a(\mathbf{u}_h, \boldsymbol{\varphi}) \\
&\quad = (\mathbf{f}, \boldsymbol{\varphi}) - (\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}) + (p_h \underline{\boldsymbol{I}}, \nabla \boldsymbol{\varphi}) + (\underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{\varphi}) - (\underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{\varphi}) \\
&\quad = (\mathbf{f} + \nabla \cdot \underline{\boldsymbol{\sigma}}_h, \boldsymbol{\varphi}) - (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{\varphi}).
\end{aligned}$$

For $T \in \mathcal{T}_h$, let $\boldsymbol{\varphi}_T^i := (\boldsymbol{\varphi}, \mathbf{e}_i)_T / |T|$, $i = 1, \ldots, d$. Then, using the assumption (4.3), the Cauchy–Schwarz inequality, the Poincaré inequality (3.2), and the definition (4.1), we get

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \boldsymbol{\varphi})_T = (\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \boldsymbol{\varphi} - \boldsymbol{\varphi}_T)_T \leq \eta_{\mathrm{R},T} \|\nabla \boldsymbol{\varphi}\|_T.$$

Next, the estimate

$$(\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{\varphi})_T \leq \eta_{\mathrm{DF},T} \|\nabla \boldsymbol{\varphi}\|_T$$

is immediate by the Cauchy–Schwarz inequality and definition (4.2). The above developments give

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\| \leq \sup_{\boldsymbol{\varphi} \in \mathbf{V}_0} \frac{\sum_{T \in \mathcal{T}_h} \{(\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T}) \|\nabla \boldsymbol{\varphi}\|_T\}}{\|\nabla \boldsymbol{\varphi}\|},$$

whence (4.4) follows by the Cauchy–Schwarz inequality. $\square$

5 A posteriori error estimate for general approximations

In this section we derive our main a posteriori error estimate. This estimate is valid for an approximation $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$, not necessarily the numerical solution. Note that the approximation can also be nonconforming and non-divergence-free.

The a posteriori error estimators on $T \in \mathcal{T}_h$ are defined as follows. The possible nonconformity of $\mathbf{u}_h$, i.e., the fact that $\mathbf{u}_h$ is not necessarily in $\mathbf{V}$, is estimated by the *nonconformity estimator*

$$\eta_{\mathrm{NC},T} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_T, \tag{5.1}$$

where $\mathbf{s}_h \in \mathbf{V}$ is arbitrary. Next, the *divergence estimator*, related to the divergence-free constraint (2.8), is given by

$$\eta_{\mathrm{D},T} := \frac{\|\nabla{\cdot}\mathbf{s}_h\|_T}{\beta}. \tag{5.2}$$

As in Section 4, the key for our a posteriori error estimates is to construct a flux (stress field) $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$ that is in approximate local equilibrium, i.e., satisfying (4.3). It enters in the *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T, \tag{5.3}$$

related to the possible violation of the equilibrium equation (2.7) in the approximate solution (here $C_{\mathrm{P},T}$ is the constant from the the Poincaré inequality (3.2)), and in the *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla\mathbf{s}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T, \tag{5.4}$$

related to the fact that the constitutive law (2.6) is not satisfied exactly by the approximate solution. Recall the definition (3.5) of the energy (semi-)norm.

Our main theorem is the following.

Theorem 5.1 (Estimate for general approximations) *Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution of (2.3a)–(2.3b) and let $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$ be arbitrary. Choose an arbitrary $\mathbf{s}_h \in \mathbf{V}$ and $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$ which satisfies (4.3). Then it holds*

$$\begin{aligned} &|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \\ &\le \left\{ \sum_{T\in\mathcal{T}_h} \eta^2_{\mathrm{NC},T} \right\}^{1/2} + \frac{1}{C_{\mathrm{S}}} \left\{ \sum_{T\in\mathcal{T}_h} \left\{ (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta^2_{\mathrm{D},T} \right\} \right\}^{1/2}. \end{aligned} \tag{5.5}$$

Proof By the triangle inequality we have

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \le \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\| + |||(\mathbf{u} - \mathbf{s}_h, p - p_h)|||.$$

Using the stability estimate (3.6) (note that $\mathbf{u} - \mathbf{s}_h \in \mathbf{V}$), we obtain

$$|||(\mathbf{u} - \mathbf{s}_h, p - p_h)||| \le \frac{1}{C_{\mathrm{S}}} \sup_{(\boldsymbol{\varphi},\psi)\in\mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{u} - \mathbf{s}_h, p - p_h), (\boldsymbol{\varphi}, \psi))}{|||(\boldsymbol{\varphi}, \psi)|||}.$$

Let $(\boldsymbol{\varphi}, \psi) \in \mathbf{V} \times Q$ be fixed. Employing the definitions (3.3) and (3.4), we have

$$\begin{aligned} &\mathcal{B}((\mathbf{u} - \mathbf{s}_h, p - p_h), (\boldsymbol{\varphi}, \psi)) \\ &\quad = \mathcal{B}((\mathbf{u}, p), (\boldsymbol{\varphi}, \psi)) - \mathcal{B}(\mathbf{s}_h, p_h), (\boldsymbol{\varphi}, \psi)) \\ &\quad = (\mathbf{f}, \boldsymbol{\varphi}) - (\nabla\mathbf{s}_h, \nabla\boldsymbol{\varphi}) + (\nabla{\cdot}\boldsymbol{\varphi}, p_h) + (\nabla{\cdot}\mathbf{s}_h, \psi). \end{aligned} \tag{5.6}$$

Next, using the fact that $\nabla{\cdot}\mathbf{u} = 0$, adding and subtracting $(\underline{\boldsymbol{\sigma}}_h, \nabla\boldsymbol{\varphi})$, and using the Green theorem, we get

$$\begin{aligned} &\mathcal{B}((\mathbf{u} - \mathbf{s}_h, p - p_h), (\boldsymbol{\varphi}, \psi)) \\ &\quad = (\mathbf{f}, \boldsymbol{\varphi}) - (\nabla\mathbf{s}_h, \nabla\boldsymbol{\varphi}) + (p_h\underline{\boldsymbol{I}}, \nabla\boldsymbol{\varphi}) + (\nabla{\cdot}\mathbf{s}_h, \psi) + (\underline{\boldsymbol{\sigma}}_h, \nabla\boldsymbol{\varphi}) - (\underline{\boldsymbol{\sigma}}_h, \nabla\boldsymbol{\varphi}) \\ &\quad = (\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \boldsymbol{\varphi}) - (\nabla\mathbf{s}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h, \nabla\boldsymbol{\varphi}) + (\nabla{\cdot}\mathbf{s}_h, \psi). \end{aligned}$$

We estimate the first two terms as in the proof of Theorem 4.1, using the equilibrium condition (4.3) and the Poincaré inequality (3.2). For the last term, we use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}
&\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\boldsymbol{\varphi},\psi)) \\
&\leq \sum_{T\in\mathcal{T}_h}(\eta_{\mathrm{R},T}+\eta_{\mathrm{DF},T})\|\nabla\boldsymbol{\varphi}\|_T + \frac{\beta}{\beta}\|\nabla{\cdot}\mathbf{s}_h\|\|\psi\| \\
&\leq \left\{\sum_{T\in\mathcal{T}_h}\left\{(\eta_{\mathrm{R},T}+\eta_{\mathrm{DF},T})^2+\eta_{\mathrm{D},T}^2\right\}\right\}^{1/2} |||(\boldsymbol{\varphi},\psi)|||.
\end{aligned}$$

The assertion then follows by collecting the above estimates. □

Let for $T\in\mathcal{T}_h$, $\eta_{\mathrm{R},T}$ and $\eta_{\mathrm{DF},T}$ be given by (4.1) and (4.2), respectively. Set $\mathbf{s}_h=\mathbf{u}_h$. Theorem 5.1 then gives the following additional result to Theorem 4.1 for conforming divergence-free approximations.

Corollary 5.1 (Pressure estimate for conforming divergence-free approximations) *Let $(\mathbf{u},p)\in\mathbf{V}\times Q$ be the weak solution of (2.3a)–(2.3b). Further, let $(\mathbf{u}_h,p_h)\in\mathbf{V}_0\times Q$ be arbitrary. Assume that $\underline{\boldsymbol{\sigma}}_h\in\underline{\boldsymbol{H}}(\mathrm{div},\Omega)$ satisfies (4.3). Then it holds*

$$\beta\|p-p_h\| \leq \frac{1}{C_{\mathrm{S}}}\left\{\sum_{T\in\mathcal{T}_h}(\eta_{\mathrm{R},T}+\eta_{\mathrm{DF},T})^2\right\}^{1/2}. \tag{5.7}$$

Let, for $T\in\mathcal{T}_h$, $\eta_{\mathrm{NC},T}$ and $\eta_{\mathrm{D},T}$ by given respectively by (5.1) and (5.2) and set

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T}h_T\|\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h-\nabla p_h+\mathbf{f}\|_T \tag{5.8}$$

and

$$\eta_{\mathrm{DF},T} := \|\nabla\mathbf{s}_h-\underline{\boldsymbol{\sigma}}_h\|_T. \tag{5.9}$$

In the sequel, we will also need the following modified version of Theorem 5.1.

Corollary 5.2 (An alternative version of Theorem 5.1) *Let $(\mathbf{u},p)\in\mathbf{V}\times Q$ be the weak solution of (2.3a)–(2.3b) and let $(\mathbf{u}_h,p_h)\in\mathbf{V}(\mathcal{T}_h)\times[Q\cap H^1(\Omega)]$ be arbitrary. Choose an arbitrary $\mathbf{s}_h\in\mathbf{V}$ and $\underline{\boldsymbol{\sigma}}_h\in\underline{\boldsymbol{H}}(\mathrm{div},\Omega)$ such that*

$$(\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h-\nabla p_h+\mathbf{f},\mathbf{e}_i)_T=0,\quad i=1,\ldots,d,\quad \forall T\in\mathcal{T}_h. \tag{5.10}$$

Then it holds

$$\begin{aligned}
&|||(\mathbf{u}-\mathbf{u}_h,p-p_h)||| \\
&\leq \left\{\sum_{T\in\mathcal{T}_h}\eta_{\mathrm{NC},T}^2\right\}^{1/2} + \frac{1}{C_{\mathrm{S}}}\left\{\sum_{T\in\mathcal{T}_h}\left\{(\eta_{\mathrm{R},T}+\eta_{\mathrm{DF},T})^2+\eta_{\mathrm{D},T}^2\right\}\right\}^{1/2}.
\end{aligned} \tag{5.11}$$

Proof We proceed as in the proof of Theorem 5.1; only the term $(\nabla{\cdot}\boldsymbol{\varphi},p_h)$ in (5.6) is treated differently. By the assumption $p_h\in H^1(\Omega)$ and the Green theorem, we get $(\nabla{\cdot}\boldsymbol{\varphi},p_h)=-(\nabla p_h,\boldsymbol{\varphi})$. The rest of the proof follows easily while using assumption (5.10) instead of (4.3). □

Remark 5.1 (Equilibrated flux $\underline{\boldsymbol{\sigma}}_h$) The equilibrated flux $\underline{\boldsymbol{\sigma}}_h$ in Theorems 4.1 and 5.1 and in Corollary 5.1 is a $\underline{\boldsymbol{H}}(\mathrm{div},\Omega)$-conforming reconstruction of the flux $\nabla\mathbf{u}_h-p_h\,\underline{\boldsymbol{I}}$. It is related to the decomposition (2.6)–(2.7). It will typically apply to such numerical methods where $p_h\notin H^1(\Omega)$. The equilibrated flux $\underline{\boldsymbol{\sigma}}_h$ in Corollary 5.2 is instead a $\underline{\boldsymbol{H}}(\mathrm{div},\Omega)$-conforming reconstruction of the flux $\nabla\mathbf{u}_h$. It is related to the decomposition (2.9)–(2.10). It will typically apply to such numerical methods where $p_h\in H^1(\Omega)$.

6 Local efficiencies

In this section, we prove the local efficiencies of the estimates introduced above.

First, we make the following assumption. Note that this assumption is only needed in this section.

Assumption 6.1 (Local efficiency) *We suppose that, for some $k \geq 1$,*

- $\mathbf{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^d$, $p_h \in \mathbb{P}_k(\mathcal{T}_h)$, *and* $\mathbf{f} \in [\mathbb{P}_k(\mathcal{T}_h)]^d$,
- *there exists a shape-regular matching simplicial submesh $\mathcal{S}_h$ of $\mathcal{T}_h$,*
- *the reconstructed flux $\underline{\boldsymbol{\sigma}}_h \in [\mathbb{P}_k(\mathcal{S}_h)]^{d\times d}$.*

When $\mathcal{T}_h$ is itself simplicial and matching, we will in many cases simply use $\mathcal{S}_h = \mathcal{T}_h$. A mesh $\mathcal{S}_h \neq \mathcal{T}_h$ will be needed for conforming methods or when $\mathcal{T}_h$ is not a simplicial mesh or is nonmatching.

We next introduce some new notation. We use $A \lesssim B$ when there exists a positive constant C, independent of mesh size, of Ω, and of $\mathbf{u}$ and p but dependent on the space dimension d, on the shape regularity parameter of the mesh $\mathcal{S}_h$, and on the maximal polynomial degree k, such that $A \leq CB$.

In order to proceed without specifying a particular numerical method, we will now make the following additional assumption. In Section 7 below, this assumption will be verified for the methods in question. Recalling, for $T \in \mathcal{T}_h$, the classical local residual error indicator (cf. [33,47])

$$\begin{aligned}\eta_{\mathrm{res},T}^2 := &\sum_{T\in\mathfrak{T}_T} \left\{h_T^2\|\mathbf{f}+\Delta\mathbf{u}_h-\nabla p_h\|_T^2+\|\nabla\cdot\mathbf{u}_h\|_T^2\right\}\\ &+\sum_{F\in\mathfrak{F}_T^{\mathrm{int}}} h_F\|[\![(\nabla\mathbf{u}_h-p_h\underline{\boldsymbol{I}})\mathbf{n}_F]\!]\|_F^2+\sum_{F\in\mathfrak{F}_T} h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F^2,\end{aligned} \tag{6.1}$$

the assumption is.

Assumption 6.2 (Approximation property) *For all $T \in \mathcal{T}_h$, there holds*

$$\|\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \tag{6.2}$$

in the case where $\underline{\boldsymbol{\sigma}}_h$ satisfies (4.3) and

$$\|\nabla\mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \tag{6.3}$$

in the case where $\underline{\boldsymbol{\sigma}}_h$ satisfies (5.10).

By $\mathcal{I}_{\mathrm{av}} : [\mathbb{P}_k(\mathcal{S}_h)]^d \to [\mathbb{P}_k(\mathcal{S}_h)]^d \cap \mathbf{V}$ we denote the operator averaging the values at each degree of freedom inside Ω and setting $\mathbf{0}$ on $\partial\Omega$. For the analysis we need the following result [1,35,19,52].

Lemma 6.3 (Averaging approximation estimate) *For $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$, there holds, for all $T \in \mathcal{T}_h$,*

$$\|\nabla(\mathbf{u}_h-\mathbf{s}_h)\|_T \lesssim \left\{\sum_{F\in\mathfrak{F}_T} h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F^2\right\}^{1/2}, \tag{6.4a}$$

$$\|\mathbf{u}_h-\mathbf{s}_h\|_T \lesssim \left\{\sum_{F\in\mathfrak{F}_T} h_F\|[\![\mathbf{u}_h]\!]\|_F^2\right\}^{1/2}. \tag{6.4b}$$

We now state and prove the main result of this section.

Theorem 6.1 (Local efficiency) *Let Assumptions 6.1 and 6.2 be satisfied. Let $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$ and let, for $T \in \mathcal{T}_h$, any of the following possibilities hold:*

- *$\eta_{\mathrm{R},T}$ and $\eta_{\mathrm{DF},T}$ are given by (4.1)–(4.2),*
- *$\eta_{\mathrm{NC},T}$, $\eta_{\mathrm{D},T}$, $\eta_{\mathrm{R},T}$, and $\eta_{\mathrm{DF},T}$ are given by (5.1)–(5.4),*
- *$\eta_{\mathrm{NC},T}$ and $\eta_{\mathrm{D},T}$ are given by (5.1)–(5.2) and $\eta_{\mathrm{R},T}$ and $\eta_{\mathrm{DF},T}$ are given by (5.8)–(5.9).*

Let finally $(\mathbf{u},p)\in\mathbf{V}\times Q$ *be the weak solution of* (2.3a)–(2.3b). *Then it holds*

$$\eta_T \lesssim |||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||_{\mathfrak{T}_T} + \left\{\sum_{F\in\mathfrak{F}_T} h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F^2\right\}^{1/2}$$

for all the local estimators $\eta_T = \eta_{\mathrm{NC},T},\ \eta_{\mathrm{D},T},\ \eta_{\mathrm{R},T},$ *and* $\eta_{\mathrm{DF},T}$.

Proof Let $T\in\mathcal{T}_h$. We will first bound the individual estimators by $\eta_{\mathrm{res},T}$ or by its components.

For $\eta_{\mathrm{DF},T}$ given by (4.2), we have $\eta_{\mathrm{DF},T}\lesssim\eta_{\mathrm{res},T}$ by Assumption 6.2. For $\eta_{\mathrm{DF},T}$ given by (5.4), the triangle inequality gives

$$\eta_{\mathrm{DF},T} \le \|\nabla\mathbf{s}_h - \nabla\mathbf{u}_h\|_T + \|\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T,$$

whence $\eta_{\mathrm{DF},T}\lesssim\eta_{\mathrm{res},T}$ by combining Assumption 6.2 and (6.4a). For the third alternative, $\eta_{\mathrm{DF},T}$ given by (5.9), using the triangle inequality,

$$\eta_{\mathrm{DF},T} \le \|\nabla\mathbf{s}_h - \nabla\mathbf{u}_h\|_T + \|\nabla\mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_T,$$

whence once again $\eta_{\mathrm{DF},T}\lesssim\eta_{\mathrm{res},T}$ by Assumption 6.2 and (6.4a).

The estimator $\eta_{\mathrm{NC},T}$ is bounded directly by (6.4a).

For the estimator $\eta_{\mathrm{R},T}$ of (4.1), we have

$$\begin{aligned}\eta_{\mathrm{R},T} &\lesssim h_T\|\mathbf{f}+\Delta\mathbf{u}_h-\nabla p_h\|_T + h_T\|-\Delta\mathbf{u}_h+\nabla p_h+\nabla\cdot\underline{\boldsymbol{\sigma}}_h\|_T\\ &= h_T\|\mathbf{f}+\Delta\mathbf{u}_h-\nabla p_h\|_T + h_T\|\nabla\cdot(\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}}-\underline{\boldsymbol{\sigma}}_h)\|_T\\ &\lesssim h_T\|\mathbf{f}+\Delta\mathbf{u}_h-\nabla p_h\|_T + \|\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}}-\underline{\boldsymbol{\sigma}}_h\|_T\end{aligned}$$

by the triangle inequality and by the inverse inequality. The bound $\eta_{\mathrm{R},T}\lesssim\eta_{\mathrm{res},T}$ thus follows by Assumption 6.2. For $\eta_{\mathrm{R},T}$ given by (5.8), we similarly have

$$\begin{aligned}\eta_{\mathrm{R},T} &\lesssim h_T\|\mathbf{f}+\Delta\mathbf{u}_h-\nabla p_h\|_T + h_T\|-\Delta\mathbf{u}_h+\nabla\cdot\underline{\boldsymbol{\sigma}}_h\|_T\\ &\lesssim h_T\|\mathbf{f}+\Delta\mathbf{u}_h-\nabla p_h\|_T + \|\nabla\mathbf{u}_h-\underline{\boldsymbol{\sigma}}_h\|_T,\end{aligned}$$

whence $\eta_{\mathrm{R},T}\lesssim\eta_{\mathrm{res},T}$ by Assumption 6.2.

We are left with bounding $\eta_{\mathrm{D},T}$. We have

$$\eta_{\mathrm{D},T} \le \frac{1}{\beta}(\|\nabla\cdot(\mathbf{s}_h-\mathbf{u}_h)\|_T + \|\nabla\cdot\mathbf{u}_h\|_T) \lesssim \frac{1}{\beta}(h_T^{-1}\|\mathbf{s}_h-\mathbf{u}_h\|_T + \|\nabla\cdot\mathbf{u}_h\|_T),$$

whence, by (6.4b), $\eta_{\mathrm{D},T}\lesssim\eta_{\mathrm{res},T}$.

We have now bounded all the local error indicators by $\eta_{\mathrm{res},T}$. The assertion of the theorem follows by the fact that this classical residual a posteriori error estimate is a lower bound for the energy error, up to the term $\{\sum_{F\in\mathfrak{F}_T} h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F^2\}^{1/2}$, cf., e.g., [47,48,22]. □

Remark 6.1 (The jump seminorm in Theorem 6.1) For conforming approximations, i.e., $\mathbf{u}_h\in\mathbf{V}$, $[\![\mathbf{u}_h]\!]=0$ and the jump seminorm contribution

$$\left\{\sum_{F\in\mathfrak{F}_T} h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F^2\right\}^{1/2}$$

vanishes. Consequently, we have the global upper and local lower bounds in the energy norm. In order to obtain both-sided estimates in the same (semi-)norm when $\mathbf{u}_h\notin\mathbf{V}$, several options are possible. Most easily, noticing that

$$\|[\![\mathbf{u}_h]\!]\|_F = \|[\![\mathbf{u}-\mathbf{u}_h]\!]\|_F, \quad F\in\partial\mathcal{T}_h,$$

we can add $\{\sum_{F\in\partial\mathcal{T}_h} h_F^{-1}\|[\![\mathbf{u}-\mathbf{u}_h]\!]\|_F^2\}^{1/2}$ to both the energy (semi-)norm and the estimate as usually done in the discontinuous Galerkin method, cf., e.g. [33]. Alternatively, when $\langle[\![\mathbf{u}_h]\!],\mathbf{e}_i\rangle_F=0$ for all

$F \in \partial\mathcal{T}_h$ and $i = 1, \ldots, d$ (this is in particular the case in the Crouzeix–Raviart nonconforming finite element method and can be achieved for a postprocessed $\tilde{\mathbf{u}}_h$ in place of $\mathbf{u}_h$ in mixed finite element methods), proceeding as in [1, Theorem 10], one can show that

$$\left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2} \lesssim \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{\mathfrak{T}_T}.$$

Finally, following [3] or [30], the jump seminorm contribution in the discontinuous Galerkin method may be bounded by the energy (semi-)norm even when the above mean value condition does not hold.

7 Application to different numerical methods

In this section, we derive a posteriori error estimates for different numerical methods using Theorem 4.1 and Corollary 5.1, Theorem 5.1, or Corollary 5.2. This consists in specifying a way for constructing the flux $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$ satisfying (4.3) or (5.10). Remark that this construction is always local. We also check, via Theorem 6.1, that the local efficiency holds for the derived estimates. This consists in verifying Assumption 6.2.

7.1 Discontinuous Galerkin method

We apply here Theorems 5.1 and 6.1 for deriving locally efficient a posteriori error estimates for the discontinuous Galerkin method. For simplicity, we suppose that $\mathcal{T}_h$ consists of simplices and is matching. The straightforward modifications to general meshes $\mathcal{T}_h$ can be carried out along the lines of [29] or [28, Appendix].

Define

$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d, \tag{7.1a}$$

$$Q_h := \mathbb{P}_{k-1}(\mathcal{T}_h) \cap Q, \tag{7.1b}$$

$k \geq 1$. Next, set

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) := &\sum_{T \in \mathcal{T}_h} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_T \; + \sum_{F \in \partial\mathcal{T}_h} \gamma_F h_F^{-1} \langle [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F \\ &- \sum_{F \in \partial\mathcal{T}_h} \left\{ \langle \{\!\{\nabla \mathbf{u}_h\}\!\} \mathbf{n}_F, [\![\mathbf{v}_h]\!] \rangle_F + \theta \langle \{\!\{\nabla \mathbf{v}_h\}\!\} \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F \right\} \end{aligned} \tag{7.2}$$

and

$$b_h(\mathbf{v}_h, q_h) := -\sum_{T \in \mathcal{T}_h} (q_h, \nabla \cdot \mathbf{v}_h)_T + \sum_{F \in \partial\mathcal{T}_h} \langle \{\!\{q_h\}\!\}, [\![\mathbf{v}_h]\!] \cdot \mathbf{n}_F \rangle_F. \tag{7.3}$$

Here, $\gamma_F > 0, F \in \partial\mathcal{T}_h$, is a parameter (chosen sufficiency large), and $\theta = \{-1, 0, 1\}$. The discontinuous Galerkin method for the problem (2.3a)–(2.3b) reads: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{7.4a}$$

$$b_h(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{7.4b}$$

We now specify $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$ satisfying (4.3). We follow [36, 27] in the second-order elliptic setting. For a recent similar reconstruction for the Stokes problem, we refer to [10]. Our postprocessed flux $\underline{\boldsymbol{\sigma}}_h$ will belong to the Raviart–Thomas–Nédélec space of tensor functions,

$$\underline{\boldsymbol{\Sigma}}^l(\mathcal{T}_h) = \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega);\, \underline{\boldsymbol{v}}_h|_T \in \underline{\boldsymbol{\Sigma}}^l(T) \quad \forall T \in \mathcal{T}_h \right\}, \tag{7.5}$$

where l is either $k-1$ or k and

$$\underline{\boldsymbol{\Sigma}}^l(T) = [\mathbb{P}_l(T)]^{d\times d} + [\mathbb{P}_l(T)]^d \otimes \mathbf{x}.$$

In particular, $\underline{\boldsymbol{\upsilon}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{T}_h)$ is such that $\nabla\cdot\underline{\boldsymbol{\upsilon}}_h|_T \in [\mathbb{P}_l(T)]^d$ for all $T \in \mathcal{T}_h$, $\underline{\boldsymbol{\upsilon}}_h \mathbf{n}_F \in [\mathbb{P}_l(F)]^d$ for all $F \in \mathcal{F}_T$ and all $T \in \mathcal{T}_h$, and such that its normal trace is continuous, cf. [17].

We prescribe $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{T}_h)$ locally on all $T \in \mathcal{T}_h$ as follows: for all $F \in \mathcal{F}_T$ and all $\mathbf{q}_h \in [\mathbb{P}_l(F)]^d$,

$$\langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_F, \mathbf{q}_h \rangle_F = \langle \{\!\{ \nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} \}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} [\![\mathbf{u}_h]\!], \mathbf{q}_h \rangle_F, \tag{7.6}$$

and for all $\underline{\boldsymbol{\tau}}_h \in [\mathbb{P}_{l-1}(T)]^{d\times d}$,

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_T = (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}, \underline{\boldsymbol{\tau}}_h)_T - \theta \sum_{F\in\mathcal{F}_T} \langle \omega_F \underline{\boldsymbol{\tau}}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F, \tag{7.7}$$

where $\omega_F := \frac{1}{2}$ for $F \in \partial\mathcal{T}_h^{\mathrm{int}}$ and $\omega_F := 1$ for $F \in \partial\mathcal{T}_h^{\mathrm{ext}}$. Observe that the quantities prescribing the moments of $\underline{\boldsymbol{\sigma}}_h \mathbf{n}_F$ are uniquely defined for each side $F \in \partial\mathcal{T}_h$, whence the continuity of the normal trace of $\underline{\boldsymbol{\sigma}}_h$. The two following lemmas are of paramount importance, implying (4.3) and (6.2), respectively.

Lemma 7.1 (Reconstructed flux in the discontinuous Galerkin method) *For $T \in \mathcal{T}_h$, let $\underline{\boldsymbol{\sigma}}_h$ be defined by (7.6)–(7.7). Then, there holds*

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{v}_h)_T = 0 \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d, \tag{7.8}$$

i.e.,

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = -(\boldsymbol{\Pi}_l \mathbf{f})|_T, \tag{7.9}$$

where $\boldsymbol{\Pi}_l$ is the $\mathbf{L}^2$-orthogonal projection onto $[\mathbb{P}_l(\mathcal{T}_h)]^d$. Thus, in particular, (4.3) holds.

Proof Let $T \in \mathcal{T}_h$ and let $\mathbf{v}_h \in [\mathbb{P}_l(T)]^d$. Owing to the Green theorem, it holds

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h)_T = -(\underline{\boldsymbol{\sigma}}_h, \nabla \mathbf{v}_h)_T + \sum_{F\in\mathcal{F}_T} \langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_T, \mathbf{v}_h \rangle_F =: T_1 + T_2.$$

Since $\nabla \mathbf{v}_h|_T \in [\mathbb{P}_{l-1}(T)]^{d\times d}$, using (7.7) yields

$$T_1 = -(\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}, \nabla \mathbf{v}_h)_T + \theta \sum_{F\in\mathcal{F}_T} \langle \omega_F \nabla \mathbf{v}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F.$$

Furthermore, the fact that $\mathbf{v}_h|_F \in [\mathbb{P}_l(F)]^d$ for all $F \in \mathcal{F}_T$ and (7.6) yield

$$T_2 = \sum_{F\in\mathcal{F}_T} \langle \{\!\{ \nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} \}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} [\![\mathbf{u}_h]\!], \mathbf{n}_T\cdot\mathbf{n}_F \mathbf{v}_h \rangle_F.$$

Extend $\mathbf{v}_h$ by $\mathbf{0}$ outside of T. Using the above identities, (7.2), (7.3), and (7.4a) yields

$$T_1 + T_2 = -a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) = -(\mathbf{f}, \mathbf{v}_h)_T,$$

whence (7.8) is valid. Finally, (7.9) results from (7.8) and the fact that $\nabla\cdot\underline{\boldsymbol{\sigma}}_h|_T \in [\mathbb{P}_l(T)]^d$. □

Lemma 7.2 (Approximation property of the reconstructed flux in the discontinuous Galerkin method) *For $T \in \mathcal{T}_h$, let $\underline{\boldsymbol{\sigma}}_h$ be defined by (7.6)–(7.7). Then (6.2) holds.*

Proof The proof follows the lines of that in the case of second-order elliptic equations. Recall that in the present case ($\mathcal{S}_h = \mathcal{T}_h$), $\mathcal{T}_h$ is shape-regular by Assumption 6.1. Using the equivalence of norms on finite-dimensional spaces, the Piola transformation, and scaling arguments, one shows that for all $T \in \mathcal{T}_h$ and all $\underline{\boldsymbol{\upsilon}}_h \in \underline{\boldsymbol{\Sigma}}^l(T)$

$$\|\underline{\boldsymbol{\upsilon}}_h\|_T^2 \lesssim \left\{ h_T \sum_{F\in\mathcal{F}_T} \|\underline{\boldsymbol{\upsilon}}_h \mathbf{n}_F\|_F^2 + \left(\sup_{\underline{\boldsymbol{\tau}}_h \in [\mathbb{P}_{l-1}(T)]^{d\times d}} \frac{(\underline{\boldsymbol{\upsilon}}_h, \underline{\boldsymbol{\tau}}_h)_T}{\|\underline{\boldsymbol{\tau}}_h\|_T} \right)^2 \right\}. \tag{7.10}$$

Define $\underline{\boldsymbol{\upsilon}}_h := \nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h$. Then, using (7.7) and the Cauchy–Schwarz and inverse inequalities, we get

$$(\underline{\boldsymbol{\upsilon}}_h, \underline{\boldsymbol{\tau}}_h)_T = \theta \sum_{F\in\mathcal{F}_T} \langle \omega_F \underline{\boldsymbol{\tau}}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!]\rangle_F \lesssim |\theta| h_T^{-1/2} \|\underline{\boldsymbol{\tau}}_h\|_T \sum_{F\in\mathcal{F}_T} \|[\![\mathbf{u}_h]\!]\|_F.$$

Note that (7.6) gives

$$\underline{\boldsymbol{\sigma}}_h \mathbf{n}_F|_F = \{\!\{\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}\}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} \boldsymbol{\Pi}_l([\![\mathbf{u}_h]\!]).$$

Thus, using (7.10) and the above developments, we have

$$\begin{aligned} \|\underline{\boldsymbol{\upsilon}}_h\|_T^2 \lesssim \Bigg\{ & h_T \sum_{F\in\mathcal{F}_T^{\mathrm{int}}} \|[\![\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}]\!] \mathbf{n}_F\|_F^2 + h_T \sum_{F\in\mathcal{F}_T} \|\gamma_F h_F^{-1} \boldsymbol{\Pi}_l([\![\mathbf{u}_h]\!])\|_F^2 \\ & + |\theta|^2 h_T^{-1} \sum_{F\in\mathcal{F}_T} \|[\![\mathbf{u}_h]\!]\|_F^2 \Bigg\}, \end{aligned}$$

whence (6.2) follows. $\square$

7.2 Conforming and conforming stabilized methods

We will show here how locally efficient a posteriori error estimates can be obtained for conforming and conforming stabilized methods using Corollary 5.2 and Theorem 6.1. We suppose that $\mathcal{T}_h$ consists of simplices and is matching. In this section, $\mathbf{V}_h \subset \mathbf{V}$, so that we systematically set $\mathbf{s}_h = \mathbf{u}_h$ throughout this section.

The conforming methods for the problem (2.3a)–(2.3b) that we consider read: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{7.11a}$$
$$b(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{7.11b}$$

In Figures 7.1–7.2, we illustrate by $\bullet$ the velocity degrees of freedom and by $\Box$ the pressure degrees of freedom. In particular, we consider the Taylor–Hood family [46,16], where, for $k \geq 1$,

$$\mathbf{V}_h = [\mathbb{P}_{k+1}(\mathcal{T}_h)]^d \cap \mathbf{V}, \quad Q_h = \mathbb{P}_k(\mathcal{T}_h) \cap C(\Omega) \cap Q.$$

The mini element [7], where

$$\mathbf{V}_h := [\mathbb{P}_1^{\mathrm{b}}(\mathcal{T}_h)]^d \cap \mathbf{V}, \quad Q_h = \mathbb{P}_1(\mathcal{T}_h) \cap C(\Omega) \cap Q,$$

where $\mathbb{P}_1^{\mathrm{b}}(\mathcal{T}_h)$ stands for $\mathbb{P}_1(\mathcal{T}_h)$ enriched by bubbles, is likewise considered. We also include the lowest-order methods, namely the cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element [40], where

$$\mathbf{V}_h := [\mathbb{P}_1(\mathcal{T}_h^{\mathrm{c}})]^d \cap \mathbf{V}, \quad Q_h = \mathbb{P}_1(\mathcal{T}_h) \cap C(\Omega) \cap Q,$$

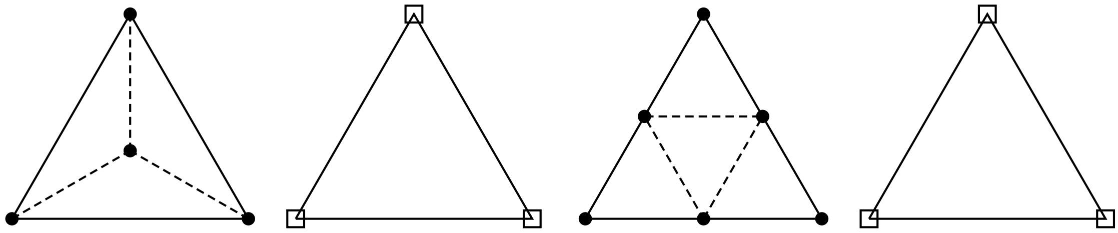

Fig. 7.1 Cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ (left) and $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ (right) conforming finite elements

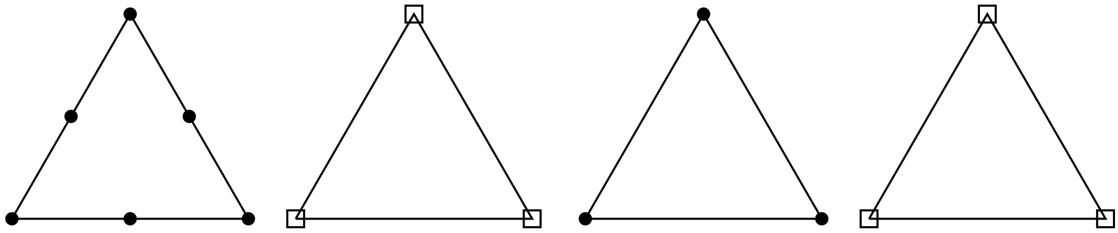

Fig. 7.2 $\mathbb{P}_2$–$\mathbb{P}_1$ Taylor–Hood conforming finite elements (left) and $\mathbb{P}_1$–$\mathbb{P}_1$ stabilized conforming finite elements (right)

with $\mathcal{T}_h^{\mathrm{c}}$ formed from $\mathcal{T}_h$ as indicated in the left part of Figure 7.1 and the $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element [12], where

$$\mathbf{V}_h := [\mathbb{P}_1(\mathcal{T}_{h/2})]^d \cap \mathbf{V}, \quad Q_h = \mathbb{P}_1(\mathcal{T}_h) \cap C(\Omega) \cap Q,$$

with $\mathcal{T}_{h/2}$ formed from $\mathcal{T}_h$ as indicated in the right part of Figure 7.1.

We also consider the conforming stabilized methods written in the general form: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{7.12a}$$
$$s_h(\mathbf{u}_h, p_h; q_h) + b(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h, \tag{7.12b}$$

where

$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d \cap \mathbf{V}, \quad Q_h = \mathbb{P}_k(\mathcal{T}_h) \cap C(\Omega) \cap Q,$$

$k \geq 1$. Let $\delta > 0$ be a parameter. In particular, we consider the Brezzi–Pitkäranta stabilized method [18], where

$$s_h(\mathbf{u}_h, p_h; q_h) = -\delta \sum_{T \in \mathcal{T}_h} h_T^2 (\nabla p_h, \nabla q_h),$$

the Hughes–Franca–Balestra stabilized method [34], where

$$s_h(\mathbf{u}_h, p_h; q_h) = \delta \sum_{T \in \mathcal{T}_h} h_T^2 (\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h, \nabla q_h)_T,$$

and the Brezzi–Douglas stabilized method [15], where

$$s_h(\mathbf{u}_h, p_h; q_h) = \delta \sum_{T \in \mathcal{T}_h} h_T^2 \{(\mathbf{f} - \nabla p_h, \nabla q_h)_T + \langle \Delta \mathbf{u}_h \cdot \mathbf{n}_T, q_h \rangle_{\partial T \cap \partial \Omega}\}.$$

7.2.1 Lowest-order continuous pressure elements

We consider here the lowest-order methods with the velocity and pressure spaces formed by continuous piecewise $\mathbb{P}_1$ polynomials, namely the cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element, the $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element, and all the above stabilized methods with $k = 1$. In the sequel, for the first two methods, $\mathcal{T}_h^{\mathrm{c}}$ or $\mathcal{T}_{h/2}$ is to be substituted systematically in place of $\mathcal{T}_h$. We follow the approach introduced in [38,51,53].

First, we need to introduce some more notation. Let the dual mesh $\mathcal{D}_h$ be formed around each vertex of $\mathcal{T}_h$ using the edge, elements, (and face in 3D) barycenters as indicated in the left part of Figure 7.3. Let $\mathcal{D}_h^{\mathrm{int}}$ correspond to the interior vertices and $\mathcal{D}_h^{\mathrm{ext}}$ to the boundary ones. Finally, we cut each $D \in \mathcal{D}_h$ into a simplicial mesh $\mathcal{S}_D$ as indicated in the right part of Figure 7.3; the matching simplicial submesh $\mathcal{S}_h$ of $\mathcal{T}_h$ (and of $\mathcal{D}_h$), needed in Assumption 6.1, is created by collecting the local meshes $\mathcal{S}_D$. We denote by $\mathcal{F}_D$ all the sides of a given $D \in \mathcal{D}_h$, by $\partial\mathcal{S}_h$ all the sides of $\mathcal{S}_h$, and by $\partial\mathcal{S}_h^{\mathrm{int}}$ all the interior sides of $\mathcal{S}_h$. Similarly, for $D \in \mathcal{D}_h$, we will employ the notation $\partial\mathcal{S}_D$ for all the sides of $\mathcal{S}_D$, $\partial\mathcal{S}_D^{\mathrm{int}}$ for all the interior sides of $\mathcal{S}_D$, and $\partial\mathcal{S}_D^{\mathrm{ext}}$ for all the boundary sides of $\mathcal{S}_D$. The notation introduced in Section 3.1 for the mesh $\mathcal{T}_h$ will be used in this section also for the meshes $\mathcal{D}_h$ and $\mathcal{S}_h$. For a vertex $V \in \mathcal{V}_h$, let ψ_V be the associated $\mathbb{P}_1$ finite element "hat" basis function. Let $\boldsymbol{\psi}_{V,i}$, $i = 1, \ldots, d$, be its vector variants such that $\boldsymbol{\psi}_{V,i}^i = \psi_V$, $\boldsymbol{\psi}_{V,i}^j = 0$ for $j = 1, \ldots, d$, $j \neq i$.

For a side $F \in \partial\mathcal{S}_h^{\mathrm{int}}$ such that $F \subset \partial D$ for some $D \in \mathcal{D}_h$, define the normal flux functions

$$\boldsymbol{\Upsilon}_F(\mathbf{u}_h) := (\nabla\mathbf{u}_h\mathbf{n}_F)|_F. \tag{7.13}$$

Note that all such sides lie inside some $T \in \mathcal{T}_h$, cf. Figure 7.3, so that $\nabla\mathbf{u}_h$ is indeed univalued thereon. The following important property holds for all the above-listed methods.

Lemma 7.3 (Local conservativity of lowest-order conforming methods) *Let* $\mathbf{f}$ *be piecewise constant on* $\mathcal{T}_h$ *and let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be given by* (7.11a)–(7.11b) *or by* (7.12a)–(7.12b) *for any of the spaces described above. Let* $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$ *be given by* (7.13)*. Then*

$$\begin{aligned} \sum_{F\in\mathcal{F}_D} \langle\boldsymbol{\Upsilon}_F(\mathbf{u}_h)\mathbf{n}_D\cdot\mathbf{n}_F, \mathbf{e}_i\rangle_F - (\nabla p_h, \mathbf{e}_i)_D + (\mathbf{f}, \mathbf{e}_i)_D = 0,& \\ i = 1, \ldots, d \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}.& \end{aligned} \tag{7.14}$$

Proof For a given dual volume $D \in \mathcal{D}_h^{\mathrm{int}}$ and associated vertex V, fix $i \in \{1, \ldots, d\}$ and consider $\boldsymbol{\psi}_{V,i}$ as the test function $\mathbf{v}_h$ in (7.11a) or (7.12a). Recall that the support of $\boldsymbol{\psi}_{V,i}$ is given by $\mathfrak{T}_V$, all the elements of $\mathcal{T}_h$ sharing V. Then, under the assumption that $\mathbf{f}$ is piecewise constant on $\mathcal{T}_h$,

$$(\mathbf{f}, \boldsymbol{\psi}_{V,i})_{\mathfrak{T}_V} = (\mathbf{f}, \mathbf{e}_i)_D \tag{7.15}$$

easily follows as $|D \cap T| = |T|/(d+1)$ for all $T \in \mathfrak{T}_V$, (cf., e.g., [53, Lemma 3.11]). Next, one derives

$$(\nabla\mathbf{u}_h, \nabla\boldsymbol{\psi}_{V,i})_{\mathfrak{T}_V} = -\langle\nabla\mathbf{u}_h\mathbf{n}_D, \mathbf{e}_i\rangle_{\partial D}$$

as in [8, Lemma 3] or [53, Lemma 3.8]. Thus, using (7.13),

$$(\nabla\mathbf{u}_h, \nabla\boldsymbol{\psi}_{V,i})_{\mathfrak{T}_V} = -\sum_{F\in\mathcal{F}_D} \langle\boldsymbol{\Upsilon}_F(\mathbf{u}_h)\mathbf{n}_D\cdot\mathbf{n}_F, \mathbf{e}_i\rangle_F.$$

Next, using the assumption $p_h \in \mathbb{P}_1(\mathcal{T}_h) \cap C(\Omega)$, implying $p_h \in H^1(\Omega)$, the Green theorem, and the fact that $\boldsymbol{\psi}_{V,i} = \mathbf{0}$ on $\partial\mathfrak{T}_V$, one comes to

$$b(\boldsymbol{\psi}_{V,i}, p_h) = -(\nabla\cdot\boldsymbol{\psi}_{V,i}, p_h)_{\mathfrak{T}_V} = (\boldsymbol{\psi}_{V,i}, \nabla p_h)_{\mathfrak{T}_V}.$$

The above right-hand side can still be rewritten equivalently as

$$(\boldsymbol{\psi}_{V,i}, \nabla p_h)_{\mathfrak{T}_V} = (\mathbf{e}_i, \nabla p_h)_D. \tag{7.16}$$

This follows from the fact that ∇p_h is piecewise constant on $\mathcal{T}_h$, so we can use the same arguments as for obtaining (7.15). Thus, combining the above arguments, (7.14) is implied by (7.11a) or by (7.12a). $\square$

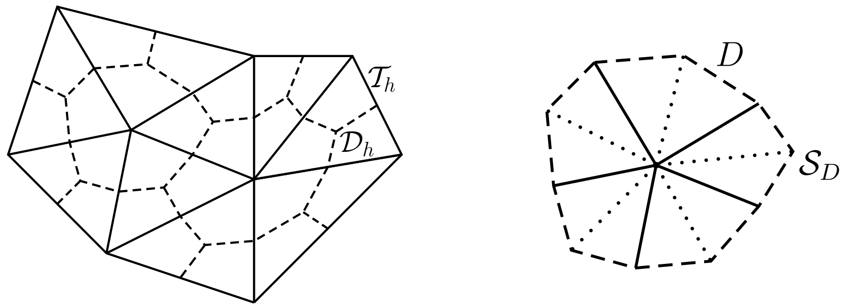

Fig. 7.3 Dual mesh $\mathcal{D}_h$ (left) and a simplicial submesh $\mathcal{S}_D$ of $D \in \mathcal{D}_h$ (right) for conforming methods in two space dimensions

Remark 7.1 (Lemma 7.3) Note that, actually, only (7.11a) or (7.12a), neither (7.11b) nor (7.12b), is needed in Lemma 7.3.

We will now define a suitable $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$; more precisely, we will construct $\underline{\boldsymbol{\sigma}}_h$ in the space $\underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_h)$, see (7.5), on the fine simplicial mesh $\mathcal{S}_h$. Prior to proceeding to a construction ensuring (5.10) (that is, a local conservation property on the mesh $\mathcal{T}_h$), let us make the following remark.

Remark 7.2 (Simple construction of $\underline{\boldsymbol{\sigma}}_h$) Following [51,53], the simplest construction of $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_h)$ is by

$$\underline{\boldsymbol{\sigma}}_h \mathbf{n}_F := \{\!\{\nabla \mathbf{u}_h \mathbf{n}_F\}\!\} \qquad \forall F \in \partial\mathcal{S}_h, \tag{7.17}$$

that is, we merely prescribe the degrees of freedom of $\underline{\boldsymbol{\sigma}}_h$ by averaging the normal components of the discontinuous approximate flux $\nabla \mathbf{u}_h$ over those sides of the mesh $\mathcal{S}_h$ which are contained in $\partial\mathcal{T}_h$ and by setting directly $\nabla \mathbf{u}_h \mathbf{n}_F$ on those sides of the mesh $\mathcal{S}_h$ which are not contained in $\partial\mathcal{T}_h$. The flux $\underline{\boldsymbol{\sigma}}_h$ defined by (7.17) (which is consistent with (7.13)) in virtue of (7.14) clearly satisfies (5.10), but on the mesh $\mathcal{D}_h^{\mathrm{int}}$ and not on the mesh $\mathcal{T}_h$. The upper bound would then needed to be written on the mesh $\mathcal{D}_h$ instead of $\mathcal{T}_h$, following [51,53]. The proof of the approximation property (6.3) is in this case straightforward: using (7.17) and (7.10), on $T \in \mathcal{S}_h$,

$$\|\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \left\{ h_F \sum_{F \in \mathcal{F}_T} \|[\![\nabla \mathbf{u}_h]\!] \mathbf{n}_F\|_F^2 \right\}^{1/2},$$

whence (6.3) follows taking into account the fact that $[\![p_h \underline{\boldsymbol{I}} \mathbf{n}_F]\!]$ is zero since $p_h \in C(\Omega)$.

Let us now define $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_h)$ such that (5.10) holds, that is, such that the local conservation property is satisfied on the original mesh $\mathcal{T}_h$. For this purpose, we adapt to the present setting the approach of [29,53]. It consists in mixed finite element solutions of local Neumann/Dirichlet problems. A local linear system on each $D \in \mathcal{D}_h$ has to be solved here but numerical experiments reveal better performance of this approach.

Let $D \in \mathcal{D}_h$ and $l \geq 0$ be given. In this section, $l = 0$, but $l \geq 1$ we will required later for higher-order conforming methods. Let $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$ be defined by (7.13). We generalize this notation to $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$, required once again later for higher-order conforming methods. Denote

$$\underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^l(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{S}_D);\, \underline{\boldsymbol{v}}_h \mathbf{n}_F = \boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h) \quad \forall F \in \partial\mathcal{S}_h^{\mathrm{int}},\, F \subset \partial D\}. \tag{7.18}$$

Let $\boldsymbol{\Pi}_l$ denote the $\mathbf{L}^2$-orthogonal projection onto $[\mathbb{P}_l(\mathcal{S}_h)]^d$. We then define $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{S}_h)$ by solving on each $D \in \mathcal{D}_h$ the following minimization problem:

$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^l(\mathcal{S}_D),\, \nabla\cdot\underline{\boldsymbol{v}}_h = \nabla p_h - \boldsymbol{\Pi}_l \mathbf{f}} \|\nabla \mathbf{u}_h - \underline{\boldsymbol{v}}_h\|_D. \tag{7.19}$$

Let $\underline{\boldsymbol{\Sigma}}^l_{\mathrm{N},0}(\mathcal{S}_D)$ be as $\underline{\boldsymbol{\Sigma}}^l_{\mathrm{N}}(\mathcal{S}_D)$ but with the normal flux condition $\underline{\boldsymbol{v}}_h\mathbf{n}_F = \mathbf{0}$ on $F \in \partial\mathcal{S}^{\mathrm{int}}_h$, $F \subset \partial D$. Let $[\mathbb{P}^*_l(\mathcal{S}_D)]^d$ be spanned by piecewise constant vectors on $\mathcal{S}_D$ with zero mean on D in each component when $D \in \mathcal{D}^{\mathrm{int}}_h$; when $D \in \mathcal{D}^{\mathrm{ext}}_h$, the mean value condition is not imposed. Then it is easy to show, cf. [29], that (7.19) is equivalent to finding $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^l_{\mathrm{N}}(\mathcal{S}_D)$ and $\mathbf{r}_h \in [\mathbb{P}^*_l(\mathcal{S}_D)]^d$ such that

$$(\underline{\boldsymbol{\sigma}}_h - \nabla\mathbf{u}_h, \underline{\boldsymbol{v}}_h)_D + (\mathbf{r}_h, \nabla\cdot\underline{\boldsymbol{v}}_h)_D = 0 \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^l_{\mathrm{N},0}(\mathcal{S}_D), \tag{7.20a}$$

$$-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \boldsymbol{\phi}_h)_D - (\mathbf{f} - \nabla p_h, \boldsymbol{\phi}_h)_D = 0 \qquad \forall \boldsymbol{\phi}_h \in [\mathbb{P}^*_l(\mathcal{S}_D)]^d. \tag{7.20b}$$

The existence and uniqueness of a solution to the above system are standard. This system is a mixed finite element approximation of a local Neumann problem on $D \in \mathcal{D}^{\mathrm{int}}_h$; the Neumann boundary conditions are given by the normal flux functions $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$. Note in particular that $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$ satisfy the Neumann compatibility condition by (7.14). When $D \in \mathcal{D}^{\mathrm{ext}}_h$, this system is a mixed finite element approximation of a local Neumann/Dirichlet problem; homogeneous Dirichlet boundary condition is prescribed on $\partial D \cap \partial\Omega$.

These developments imply.

Lemma 7.4 (Reconstructed flux in lowest-order conforming methods) *Let $\mathbf{f}$ be piecewise constant on $\mathcal{T}_h$ and let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ be given by (7.11a)–(7.11b) or by (7.12a)–(7.12b) for any of the lowest-order methods. Let $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$ be given by (7.13) and prescribe $\underline{\boldsymbol{\sigma}}_h$ by (7.19), with $l = 0$. Then (5.10) holds. More precisely,*

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = (\nabla p_h - \mathbf{f})|_T \qquad \forall T \in \mathcal{S}_h. \tag{7.21}$$

To finish this section, we now check that the approximation property (6.3) holds.

Lemma 7.5 (Approximation property of the reconstructed flux in lowest-order conforming methods) *Let the assumptions of Lemma 7.4 be verified. Then the approximation property (6.3) holds.*

Proof Let $D \in \mathcal{D}_h$ and let $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D)$ and $\mathbf{r}_h \in [\mathbb{P}^*_0(\mathcal{S}_D)]^d$ be given by (7.20a)–(7.20b). Extending the approach of [50, Section 4.1] (cf. also [6,5]) to the vector case, we define a postprocessing $\tilde{\mathbf{r}}_h$ of $\mathbf{r}_h$ such that

$$\nabla\tilde{\mathbf{r}}_h|_T = (\underline{\boldsymbol{\sigma}}_h - \nabla\mathbf{u}_h)|_T \qquad \forall T \in \mathcal{S}_D, \tag{7.22a}$$

$$\frac{(\tilde{\mathbf{r}}_h, \mathbf{e}_i)_T}{|T|} = \mathbf{r}^i_h|_T, \qquad i = 1, \ldots, d, \quad \forall T \in \mathcal{S}_D. \tag{7.22b}$$

Note that this is a cheap local procedure. It follows from (7.22a), (7.22b), and (7.20a) that

$$(\nabla\tilde{\mathbf{r}}_h, \underline{\boldsymbol{v}}_h)_D + (\tilde{\mathbf{r}}_h, \nabla\cdot\underline{\boldsymbol{v}}_h)_D = 0 \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^l_{\mathrm{N},0}(\mathcal{S}_D).$$

Fixing one $F \in \partial\mathcal{S}^{\mathrm{int}}_D$, choosing the basis functions of $\underline{\boldsymbol{\Sigma}}^l_{\mathrm{N},0}(\mathcal{S}_D)$ having nonzero normal trace only across this side, and using the Green theorem, we arrive at

$$\langle [\![\tilde{\mathbf{r}}_h]\!], \mathbf{e}_i\rangle_F = 0, \qquad i = 1, \ldots, d. \tag{7.23}$$

This means that the postprocessed $\tilde{\mathbf{r}}_h$ has the mean value of the jump in each component equal to zero on the interior sides of $\mathcal{S}_D$. Alternatively, we can say that $\tilde{\mathbf{r}}_h$ has means of traces continuous on the interior sides of $\mathcal{S}_D$.

If $D \in \mathcal{D}^{\mathrm{ext}}_h$, we arrive similarly at

$$\langle \tilde{\mathbf{r}}_h, \mathbf{e}_i\rangle_F = 0, \qquad i = 1, \ldots, d, \tag{7.24}$$

for all $F \in \partial\mathcal{S}^{\mathrm{ext}}_D$ such that $F \subset \partial\Omega$. Thus, on exterior sides of $\mathcal{S}_D$ belonging to $\partial\Omega$, the mean value of each component of $\tilde{\mathbf{r}}_h$ is zero.

Finally, for $D \in \mathcal{D}^{\mathrm{int}}_h$, we have that $(\mathbf{r}_h, \mathbf{e}_i)_D = 0$, $i = 1, \ldots, d$, from the definition of $[\mathbb{P}^*_0(\mathcal{S}_D)]^d$. From this fact and (7.22b), we deduce that

$$(\tilde{\mathbf{r}}_h, \mathbf{e}_i)_D = 0, \qquad i = 1, \ldots, d. \tag{7.25}$$

Thus, on dual volumes not touching the boundary, the mean value of each component of $\tilde{\mathbf{r}}_h$ is zero.

We denote by $\mathbf{M}(\mathcal{S}_D) \subset [\mathbb{P}_2(\mathcal{S}_D)]^d$ the corresponding space of polynomials verifying (7.23), (7.24), and (7.25). Using the above developments, we have

$$\|\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_D = \sup_{\mathbf{m}_h \in \mathbf{M}(\mathcal{S}_D),\, \|\nabla \mathbf{m}_h\|_D = 1} (\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h, \nabla \mathbf{m}_h)_D. \tag{7.26}$$

We now develop the right-hand side of (7.26). Using the Green theorem, the fact that $\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h = \nabla p_h - \mathbf{f}$ for all $T \in \mathcal{S}_D$, see (7.21), (7.23) (with $\tilde{\mathbf{r}}_h$ replaced by $\mathbf{m}_h$) and the facts that $((\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h)\mathbf{n}_T)|_F$ is in $[\mathbb{P}_0(F)]^d$ and that $[\![\underline{\boldsymbol{\sigma}}_h \mathbf{n}_F]\!]|_F = 0$ for all sides $F \in \partial\mathcal{S}_D^{\rm int}$, we arrive at

$$\begin{aligned}
&(\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h, \nabla \mathbf{m}_h)_D \\
&\quad = \sum_{T \in \mathcal{S}_D} \{-(\mathbf{m}_h, \nabla{\cdot}(\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h))_T + \langle (\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h)\mathbf{n}_T, \mathbf{m}_h\rangle_{\partial T}\} \\
&\quad = -\sum_{T \in \mathcal{S}_D} (\mathbf{m}_h, \mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h)_T + \sum_{F \in \partial\mathcal{S}_D^{\rm int}} \langle [\![\nabla \mathbf{u}_h \mathbf{n}_F]\!], \mathbf{m}_h\rangle_F.
\end{aligned} \tag{7.27}$$

We have also used that $\underline{\boldsymbol{\sigma}}_h \mathbf{n}_F = \nabla \mathbf{u}_h \mathbf{n}_F$ for all boundary sides F of $\mathcal{S}_D$ not included in $\partial\Omega$ since $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}_{\rm N}^0(\mathcal{S}_D)$, and (7.24) for all boundary sides F of $\mathcal{S}_D$ included in $\partial\Omega$. By the Cauchy–Schwarz inequality and the inverse inequality $\|\mathbf{m}_h\|_F \lesssim h_F^{-\frac{1}{2}}\|\mathbf{m}_h\|_T$, we can further estimate

$$\begin{aligned}
&(\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h, \nabla \mathbf{m}_h)_D \\
&\leq \left\{\sum_{T \in \mathcal{S}_D} h_T^{-2}\|\mathbf{m}_h\|_T^2\right\}^{1/2} \left\{\sum_{T \in \mathcal{S}_D} h_T^2\|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T^2\right\}^{1/2} \\
&\quad + \left\{\sum_{F \in \partial\mathcal{S}_D^{\rm int}} h_F^{-1}\|\mathbf{m}_h\|_F^2\right\}^{1/2} \left\{\sum_{F \in \partial\mathcal{S}_D^{\rm int}} h_F\|[\![\nabla \mathbf{u}_h \mathbf{n}_F]\!]\|_F^2\right\}^{1/2} \\
&\lesssim h_D^{-1}\|\mathbf{m}_h\|_D \left\{\sum_{T \in \mathcal{S}_D} h_T^2\|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T^2 + \sum_{F \in \partial\mathcal{S}_D^{\rm int}} h_F\|[\![\nabla \mathbf{u}_h \mathbf{n}_F]\!]\|_F^2\right\}^{1/2}.
\end{aligned}$$

Recall that, as $\mathbf{m}_h \in \mathbf{M}(\mathcal{S}_D)$, we have (7.24) or (7.25) for $\mathbf{m}_h$. Thus, the discrete Poincaré/Friedrichs inequality

$$\|\mathbf{m}_h\|_D \lesssim h_D \|\nabla \mathbf{m}_h\|_D$$

can be easily proven along the lines of [49]. Consequently, (6.3) follows from the above estimates, the fact that $[\![p_h \underline{\boldsymbol{I}} \mathbf{n}_F]\!] = 0$ for all $F \in \mathcal{S}_D^{\rm int}$ since $p_h \in C(\Omega)$, and (7.26). $\square$

7.2.2 Higher-order continuous pressure elements

The approach of the previous section does not generalize directly to higher-order conforming and conforming stabilized methods. When $\mathbf{V}_h$ contains piecewise polynomials of degree higher than 1 or $\mathbf{f}$ is not piecewise constant on $\mathcal{T}_h$, the local conservation property (7.14) does not hold. Here, we extend the approach of Section 7.2.1 to higher-order elements, namely the Taylor–Hood elements, the mini element, and higher-order stabilized methods.

Recall that each dual volume $D \in \mathcal{D}_h^{\rm int}$ is associated with one vertex $V \in \mathcal{V}_h^{\rm int}$ and recall also the definition $\boldsymbol{\psi}_{V,i}$, $i = 1, \ldots, d$, of the vector Lagrange basis functions, see Section 7.2.1. For $V \in \mathcal{V}_h$, denote by $\partial\mathfrak{T}_V^{\rm int}$ all the interior sides of the patch $\mathfrak{T}_V$. Then, choosing $\mathbf{v}_h = \boldsymbol{\psi}_{V,i}$, $i = 1, \ldots, d$ as the test function in (7.11a) or (7.12a) and combining with the Green theorem, we obtain the following result.

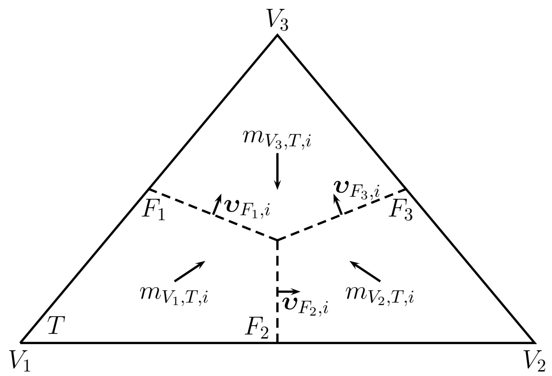

Fig. 7.4 Equilibration of the correction terms inside each triangle

Lemma 7.6 (Higher-order conforming methods on a dual mesh) *Let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be given by* (7.11a)–(7.11b) *or by* (7.12a)–(7.12b) *with* $\mathbf{V}_h \subset \mathbf{V}$ *and* $Q_h \subset H^1(\Omega)$*. Let the normal flux functions* $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$ *be given by* (7.13)*. Then*

$$\begin{aligned}\sum_{F \in \mathcal{F}_D} &\langle \boldsymbol{\Upsilon}_F(\mathbf{u}_h)\mathbf{n}_D{\cdot}\mathbf{n}_F, \mathbf{e}_i\rangle_F - (\nabla p_h, \mathbf{e}_i)_D + (\mathbf{f}, \mathbf{e}_i)_D \\ &= (\mathbf{f} + \Delta\mathbf{u}_h - \nabla p_h, \mathbf{e}_i)_D - (\mathbf{f} + \Delta\mathbf{u}_h - \nabla p_h, \boldsymbol{\psi}_{V,i})_{\mathfrak{T}_V} \\ &\quad + \sum_{F \in \partial\mathfrak{T}_V^{\mathrm{int}}} \langle [\![\nabla\mathbf{u}_h\mathbf{n}_F]\!], \boldsymbol{\psi}_{V,i}\rangle_F - \sum_{F \in \partial\mathcal{S}_D^{\mathrm{int}}} \langle [\![\nabla\mathbf{u}_h\mathbf{n}_F]\!], \mathbf{e}_i\rangle_F, \\ &\qquad i = 1, \ldots, d, \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}.\end{aligned} \tag{7.28}$$

Recall that lowest-order methods are locally conservative on the elements D of $\mathcal{D}_h^{\mathrm{int}}$, see (7.14). We can see from (7.28) that higher-order methods are also locally conservative, up to additional correction terms featuring the element residuals $\mathbf{f} + \Delta\mathbf{u}_h - \nabla p_h$ and edge residuals $[\![\nabla\mathbf{u}_h\mathbf{n}_F]\!]$. We now want to redistribute these correction terms to the normal flux functions $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$ of (7.13) to obtain new normal flux functions $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$ for which (7.14) (and not (7.28)) holds. We achieve this in a spirit similar to equilibration techniques of [4,23,14].

For $D \in \mathcal{D}_h^{\mathrm{int}}$, the associated vertex V, $T \in \mathfrak{T}_V$, and $i = 1, \ldots, d$, denote the contribution to the above correction terms by

$$\begin{aligned} &m_{V,T,i} \\ &:= -(\mathbf{f} + \Delta\mathbf{u}_h - \nabla p_h, \mathbf{e}_i)_{T \cap D} + (\mathbf{f} + \Delta\mathbf{u}_h - \nabla p_h, \boldsymbol{\psi}_{V,i})_T \\ &\quad - \frac{1}{2} \sum_{F \in \mathcal{F}_T^{\mathrm{int}}} \langle [\![\nabla\mathbf{u}_h\mathbf{n}_F]\!], \boldsymbol{\psi}_{V,i}\rangle_F + \frac{1}{2} \sum_{F \in \mathcal{F}_T^{\mathrm{int}}} \langle [\![\nabla\mathbf{u}_h\mathbf{n}_F]\!], \mathbf{e}_i\rangle_{F \cap D}.\end{aligned} \tag{7.29}$$

We will speak about these quantities as of "normal fluxes" $m_{V,T,i}$. Remark that $[\![\nabla\mathbf{u}_h\mathbf{n}_F]\!] = 0$ on such sides $F \in \partial\mathcal{S}_D^{\mathrm{int}}$ which are not contained in $\partial\mathcal{T}_h$, cf. Figure 7.3. Thus, from (7.28) and the above formula, we have

$$\begin{aligned}\sum_{F \in \mathcal{F}_D} \langle \boldsymbol{\Upsilon}_F(\mathbf{u}_h)\mathbf{n}_D{\cdot}\mathbf{n}_F, \mathbf{e}_i\rangle_F - (\nabla p_h, \mathbf{e}_i)_D + (\mathbf{f}, \mathbf{e}_i)_D + \sum_{T \in \mathfrak{T}_V} m_{V,T,i} = 0, \\ i = 1, \ldots, d, \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}.\end{aligned} \tag{7.30}$$

For the sake of simplicity, let us define $m_{V,T,i}$ in the same way also for $D \in \mathcal{D}_h^{\mathrm{ext}}$ and the associated vertex V.

Consider a fixed $T \in \mathcal{T}_h$ and $i = 1, \ldots, d$. We have associated the normal flux $m_{V_j,T,i}$ to each of the vertices V_j of T, $j = 1, \ldots, d+1$, cf. Figure 7.4. We now want to equilibrate the normal fluxes $m_{V_j,T,i}$:

the purpose is to associate to each of the sides $F_m \subset T$, $m = 1, \ldots, d+1$, $F_m \in \partial\mathcal{S}_h^{\mathrm{int}}$ such that $F \subset \partial D$ for some $D \in \mathcal{D}_h$, a correction normal flux $\upsilon_{F_m,i}$ (in the direction of the fixed normal $\mathbf{n}_F$) such that the following holds (we give an example for $d = 2$, corresponding to Figure 7.4):

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \upsilon_{F_1,i} \\ \upsilon_{F_2,i} \\ \upsilon_{F_3,i} \end{pmatrix} = \begin{pmatrix} m_{V_1,T,i} \\ m_{V_2,T,i} \\ m_{V_3,T,i} \end{pmatrix}. \tag{7.31}$$

The value $m_{V_1,T,i}$ represents the total normal flux from the element $T \cap D_1$ to the elements $T \cap D_2$ and $T \cap D_3$ (where D_i are the dual volumes associated with the vertices V_i). We clearly want to keep this total normal flux but to split it into the side normal fluxes $\upsilon_{F_1,i}$ and $\upsilon_{F_2,i}$; we proceed similarly for $m_{V_2,T,i}$ and $m_{V_3,T,i}$. The essential feature is that the corrections normal fluxes $\upsilon_{F_m,i}$ are univocally defined for each side F_m, $m = 1, \ldots, d+1$, cf. once again Figure 7.4.

It turns out that the system matrix in (7.31) is singular, as the sum of all the row vectors equals zero. It is, however, easy to check that its rank is equal to d. Fortunately, the right-hand side in (7.31) is compatible: by the fact that the basis functions $\boldsymbol{\psi}_{V_j,i}$ form a partition of unity on the chosen element $T \in \mathcal{T}_h$,

$$\sum_{j=1}^{d+1} \boldsymbol{\psi}_{V_j,i}|_T = \mathbf{e}_i|_T,$$

we easily get

$$\sum_{j=1}^{d+1} m_{V_j,T,i} = 0,$$

$i = 1, \ldots, d$. Thus, there exists a solution to (7.31). Note that (7.31) is always a system of a fixed small size $(d+1) \times (d+1)$ on each $T \in \mathcal{T}_h$, for approximations (7.11a)–(7.11b) or (7.12a)–(7.12b) of any order k.

Using $\upsilon_{F_m,i}$ for each $T \in \mathcal{T}_h$, we can now define new normal flux functions $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$ for sides $F \in \partial\mathcal{S}_h^{\mathrm{int}}$ such that $F \subset \partial D$ for some $D \in \mathcal{D}_h$, in a way that (7.14) holds. More precisely, let

$$(\boldsymbol{\upsilon}_F(\mathbf{u}_h, p_h))^i := |F|^{-1} \upsilon_{F,i}, \qquad i = 1, \ldots, d. \tag{7.32}$$

Note that, consequently, (7.31) gives

$$\sum_{T \in \mathfrak{T}_V} m_{V,T,i} = \sum_{F \in \mathcal{F}_D} \upsilon_{F,i} \mathbf{n}_D \cdot \mathbf{n}_F = \sum_{F \in \mathcal{F}_D} \langle \boldsymbol{\upsilon}_F(\mathbf{u}_h, p_h) \mathbf{n}_D \cdot \mathbf{n}_F, \mathbf{e}_i \rangle_F \tag{7.33}$$

for every $D \in \mathcal{D}_h^{\mathrm{int}}$ and the associated vertex V, $i = 1, \ldots, d$. Let $F \in \partial\mathcal{S}_h^{\mathrm{int}}$ such that $F \subset \partial D$ for some $D \in \mathcal{D}_h$ and set

$$\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h) := (\nabla \mathbf{u}_h \mathbf{n}_F)|_F + \boldsymbol{\upsilon}_F(\mathbf{u}_h, p_h). \tag{7.34}$$

We then see that (7.30) together with (7.33) and (7.34) implies (7.14). Recall the definition of the space $\underline{\boldsymbol{\Sigma}}^l(\mathcal{S}_D)$ by (7.5). For the new normal flux functions $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$ of (7.34), we can define the space $\underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^l(\mathcal{S}_D)$ by (7.18). We then set $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{S}_h)$ by (7.19) or, equivalently, by (7.20a)–(7.20b), where we put $l = k-1$. Note that the equation (7.14) holds for the new normal flux functions $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$, whence the local Neumann problems are well-posed for $D \in \mathcal{D}_h^{\mathrm{int}}$. As in Lemma 7.4, condition (5.10) is satisfied by this definition of $\underline{\boldsymbol{\sigma}}_h$.

The last point consists in proving the approximation property (6.3). We proceed for this purpose as in Lemma 7.5. Let $D \in \mathcal{D}_h$. Firstly, we need to replace the definition (7.22a)–(7.22b), valid in the lowest-order case, by defining $\tilde{\mathbf{r}}_h \in \mathbf{M}(\mathcal{S}_D)$ by

$$\underline{\boldsymbol{\Pi}}_{\underline{\boldsymbol{\Sigma}}_h(T)}(\nabla \tilde{\mathbf{r}}_h|_T) = (\underline{\boldsymbol{\sigma}}_h - \nabla \mathbf{u}_h)|_T \qquad \forall T \in \mathcal{S}_D, \tag{7.35a}$$

$$\boldsymbol{\Pi}_{\mathbf{V}_h(T)}(\tilde{\mathbf{r}}_h|_T) = \mathbf{r}_h|_T \qquad \forall T \in \mathcal{S}_D. \tag{7.35b}$$

Here $\boldsymbol{\mathit{\Pi}}_{\underline{\boldsymbol{\Sigma}}_h(T)}$ is the $\underline{\boldsymbol{L}}^2$-orthogonal projection onto $\underline{\boldsymbol{\Sigma}}_h(T)$ and $\boldsymbol{\Pi}_{\mathbf{V}_h(T)}$ is the $\mathbf{L}^2$-orthogonal projection onto $\mathbf{V}_h(T)$. The space $\mathbf{M}(\mathcal{S}_D)$ is a vector variant of that of [6,5]. What is important in the present analysis is that by (7.35a)–(7.35b), the properties (7.23), (7.24), and (7.25) still hold. More precisely, the orthogonality in (7.23) and (7.24) holds up to polynomials of order l in each component and not only for constants in each component. Similarly to (7.26), we now have

$$\|\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_D \lesssim \sup_{\mathbf{m}_h \in \mathbf{M}(\mathcal{S}_D),\, \|\nabla \mathbf{m}_h\|_D = 1} (\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h, \nabla \mathbf{m}_h)_D,$$

using [54, Lemma 5.4]. Suppose now for simplicity that $\mathbf{f} \in [\mathbb{P}_l(\mathcal{S}_h)]^d$ instead of $\mathbf{f} \in [\mathbb{P}_k(\mathcal{S}_h)]^d$ required in Assumption 6.1. Then (7.27) still holds but with an additional factor

$$-\sum_{F \in \partial \mathcal{S}_D^{\mathrm{ext}},\, F \not\subset \partial \Omega} \langle \boldsymbol{\upsilon}_F(\mathbf{u}_h, p_h) \mathbf{n}_D {\cdot} \mathbf{n}_F, \mathbf{m}_h \rangle_F$$

in the last equality. We thus need to bound this factor, which we do by the Cauchy–Schwarz inequality and obtain

$$\Bigg\{ \sum_{F \in \partial \mathcal{S}_D^{\mathrm{ext}},\, F \not\subset \partial \Omega} h_F^{-1} \|\mathbf{m}_h\|_F^2 \Bigg\}^{1/2} \Bigg\{ \sum_{F \in \partial \mathcal{S}_D^{\mathrm{ext}},\, F \not\subset \partial \Omega} h_F \|\boldsymbol{\upsilon}_F(\mathbf{u}_h, p_h)\|_F^2 \Bigg\}^{1/2}.$$

The first term above can be treated as in Lemma 7.5 and we are left with bounding the second one.

Let $F \in \partial \mathcal{S}_h^{\mathrm{int}}$, let $T \in \mathcal{T}_h$ be such that $F \subset T$, and let V_j, D_j, $j = 1, \ldots, d+1$, be the vertices of T and the associated dual volumes. Using (7.29), (7.31), and (7.32), we arrive at

$$\begin{aligned}
&h_F^{1/2} \|\boldsymbol{\upsilon}_F(\mathbf{u}_h, p_h)\|_F \\
&\lesssim h_F^{1/2} \sum_{j=1}^{d+1} \sum_{i=1}^{d} \||F|^{-1} (\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h, \mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i})_T\|_F \\
&\quad + h_F^{1/2} \sum_{F' \in \mathcal{F}_T^{\mathrm{int}}} \sum_{j=1}^{d+1} \sum_{i=1}^{d} \||F|^{-1} \langle [\![\nabla \mathbf{u}_h \mathbf{n}_{F'}]\!], \mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i} \rangle_{F'}\|_F.
\end{aligned}$$

Let $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, d+1\}$, and $F' \in \mathcal{F}_T^{\mathrm{int}}$ be given. Then,

$$\begin{aligned}
&h_F^{1/2} \||F|^{-1} (\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h, \mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i})_T\|_F \\
&\quad = h_F^{1/2} |F|^{-1/2} |(\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h, \mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i})_T| \\
&\quad \le h_F^{1/2} |F|^{-1/2} \|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T \|\mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i}\|_T \\
&\quad \lesssim h_F^{1/2} |F|^{-1/2} \|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T |T|^{\frac{1}{2}} \\
&\quad \lesssim h_T \|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T
\end{aligned}$$

by the Cauchy–Schwarz inequality and the facts that $\|\mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i}\|_T \lesssim |T|^{\frac{1}{2}}$, $|T|^{\frac{1}{2}}/|F|^{\frac{1}{2}} \lesssim h_F^{1/2}$, and $h_F \lesssim h_T$. Similarly,

$$\begin{aligned}
&h_F^{1/2} \||F|^{-1} \langle [\![\nabla \mathbf{u}_h \mathbf{n}_{F'}]\!], \mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i} \rangle_{F'}\|_F \\
&\quad = h_F^{1/2} |F|^{-1/2} |\langle [\![\nabla \mathbf{u}_h \mathbf{n}_{F'}]\!], \mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i} \rangle_{F'}| \\
&\quad \lesssim h_F^{1/2} |F|^{-1/2} \|[\![\nabla \mathbf{u}_h \mathbf{n}_{F'}]\!]\|_{F'} \|\mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i}\|_{F'} \\
&\quad \lesssim |F|^{-1/2} \|[\![\nabla \mathbf{u}_h \mathbf{n}_{F'}]\!]\|_{F'} \|\mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i}\|_T \\
&\quad \lesssim h_{F'}^{1/2} \|[\![\nabla \mathbf{u}_h \mathbf{n}_{F'}]\!]\|_{F'},
\end{aligned}$$

employing also the inverse inequality $\|\mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i}\|_{F'} \lesssim h_{F'}^{-1/2} \|\mathbf{e}_i|_{D_j} - \boldsymbol{\psi}_{V_j,i}\|_T$. Combining all the above results, we arrive at the conclusion that the approximation property (6.3) holds in the higher-order case as well.

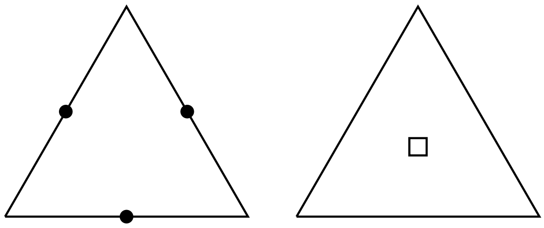

Fig. 7.5 $\mathbb{P}_1$–$\mathbb{P}_0$ nonconforming Crouzeix–Raviart finite elements

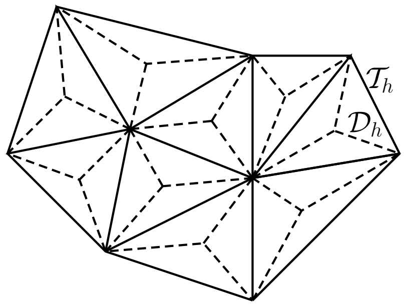

Fig. 7.6 Dual mesh $\mathcal{D}_h$ for the nonconforming Crouzeix–Raviart method in two space dimensions

Remark 7.3 (Standard equilibration techniques) The equilibration techniques of [4,23,14] can be used in order to produce $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{T}_h)$ satisfying (5.10). Under the condition that (6.3) holds, they can likewise be used in the present framework.

7.3 Nonconforming methods

Here, we derive locally efficient a posteriori error estimates for the lowest-order nonconforming Crouzeix–Raviart method using Theorems 5.1 and 6.1. We follow the approach of Section 7.2.1. Extension to higher-order methods is possible along the lines of the approach of Section 7.2.2.

Let $\mathcal{T}_h$ be simplicial and matching and let

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v}_h \in [\mathbb{P}_1(\mathcal{T}_h)]^d;\ \langle [\![\mathbf{v}_h]\!], \mathbf{e}_i\rangle_F = 0, \quad i = 1,\dots,d, \qquad \forall F \in \partial\mathcal{T}_h\},\\ Q_h &:= \mathbb{P}_0(\mathcal{T}_h) \cap Q, \end{aligned}$$

cf. Figure 7.5. The lowest-order nonconforming Crouzeix–Raviart method for the problem (2.3a)–(2.3b) reads, see [21]: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{7.36a}$$

$$b(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{7.36b}$$

Let the dual mesh $\mathcal{D}_h$ be formed around each side of $\mathcal{T}_h$ using the element barycenters as indicated in Figure 7.6; $\mathcal{D}_h^{\mathrm{int}}$ correspond to the interior sides and $\mathcal{D}_h^{\mathrm{ext}}$ to the boundary ones. For a side $F \in \partial\mathcal{T}_h$, let ψ_F be the $\mathbb{P}_1$ nonconforming finite element basis function. Let $\boldsymbol{\psi}_{F,i}$, $i = 1,\dots,d$, be its vector variants such that $\boldsymbol{\psi}_{F,i}^i = \psi_F$, $\boldsymbol{\psi}_{F,i}^j = 0$ for $j = 1,\dots,d$, $j \neq i$. We will also need the fine simplicial mesh $\mathcal{S}_h$ formed by the $d+1$ subsimplices of each $T \in \mathcal{T}_h$, cf. once again Figure 7.6.

For a side $F \in \partial\mathcal{S}_h^{\mathrm{int}}$ such that $F \subset \partial D$ for some $D \in \mathcal{D}_h$, define the normal flux functions

$$\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h) := (\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}})\mathbf{n}_F. \tag{7.37}$$

Note that, as in the conforming setting of Section 7.2, all such sides lie inside some $T \in \mathcal{T}_h$, cf. Figure 7.6, so that $\nabla\mathbf{u}_h$ and p_h are indeed univalued thereon.

As in Lemma 7.3 in the conforming case, the following important property holds.

Lemma 7.7 (Local conservativity of the nonconforming Crouzeix–Raviart method) *Let* $\mathbf{f}$ *be piecewise constant on* $\mathcal{T}_h$ *and let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be given by* (7.36a)–(7.36b). *Let* $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$ *be given by* (7.37). *Then*

$$\sum_{F\in\mathcal{F}_D} \langle \boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)\mathbf{n}_D\cdot\mathbf{n}_F, \mathbf{e}_i\rangle_F + (\mathbf{f}, \mathbf{e}_i)_D = 0, \qquad i = 1, \dots, d, \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}. \tag{7.38}$$

Proof For a given dual volume $D \in \mathcal{D}_h^{\mathrm{int}}$ and associated side F, fix $i \in \{1, \dots, d\}$ and consider $\boldsymbol{\psi}_{F,i}$ as the test function $\mathbf{v}_h$ in (7.36a). Recall that the support of $\boldsymbol{\psi}_{F,i}$ is given by the two elements sharing F, denoted by $\mathcal{T}_F$. Then, under the assumption that $\mathbf{f}$ is piecewise constant on $\mathcal{T}_h$,

$$(\mathbf{f}, \boldsymbol{\psi}_{F,i})_{\mathcal{T}_F} = (\mathbf{f}, \mathbf{e}_i)_D$$

easily follows as $|D \cap T| = |T|/(d+1)$ for all $T \in \mathcal{T}_F$. Next, consider $T \in \mathcal{T}_F$. One has

$$\begin{aligned}
(\nabla \mathbf{u}_h, \nabla \boldsymbol{\psi}_{F,i})_T &= (\nabla \mathbf{u}_h^i, \nabla \psi_F)_T = -(\Delta \mathbf{u}_h^i, \psi_F)_T + \langle \nabla \mathbf{u}_h^i\cdot\mathbf{n}_T, \psi_F\rangle_{\partial T} \\
&= \langle \nabla \mathbf{u}_h^i\cdot\mathbf{n}_T, \psi_F\rangle_F = \langle \nabla \mathbf{u}_h^i\cdot\mathbf{n}_T, 1\rangle_F \\
&= -\sum_{F'\in\mathcal{F}_D,\, F'\subset T} \langle \nabla \mathbf{u}_h^i\cdot\mathbf{n}_D, 1\rangle_{F'} \\
&= -\sum_{F'\in\mathcal{F}_D,\, F'\subset T} \langle \nabla \mathbf{u}_h\mathbf{n}_D, \mathbf{e}_i\rangle_{F'},
\end{aligned} \tag{7.39}$$

using the fact that $\Delta \mathbf{u}_h^i = 0$ as $\mathbf{u}_h|_T \in [\mathbb{P}_1(T)]^d$, the facts that $\nabla \mathbf{u}_h^i\cdot\mathbf{n}_T$ is constant on all sides $F \in \mathcal{F}_T$, that $\langle 1, \psi_F\rangle_{F'} = 0$ for $F' \in \mathcal{F}_T$, $F' \neq F$, and that $\langle 1, \psi_F\rangle_F = \langle 1, 1\rangle_F$, and finally once again the Green theorem and the fact that $\Delta \mathbf{u}_h^i = 0$. Finally,

$$\begin{aligned}
b(\boldsymbol{\psi}_{F,i}, p_h) &= -(\nabla\cdot\boldsymbol{\psi}_{F,i}, p_h)_{\mathcal{T}_F} = -(\nabla\boldsymbol{\psi}_{F,i}, p_h\underline{\boldsymbol{I}})_{\mathcal{T}_F} \\
&= -\sum_{T\in\mathcal{T}_F} (\nabla\psi_F, \nabla(p_h\mathbf{x}^i))_T = \sum_{F'\in\mathcal{F}_D,} \langle \nabla(p_h\mathbf{x}^i)\cdot\mathbf{n}_D, 1\rangle_{F'} \\
&= \sum_{F'\in\mathcal{F}_D} \langle p_h\mathbf{e}_i\cdot\mathbf{n}_D, 1\rangle_{F'} = \sum_{F'\in\mathcal{F}_D} \langle p_h\underline{\boldsymbol{I}}\mathbf{n}_D, \mathbf{e}_i\rangle_{F'}
\end{aligned}$$

by the same arguments as in (7.39) and using that $p_h \in \mathbb{P}_0(\mathcal{T}_h)$. Combining the above results, the assertion of the lemma follows. □

We will now construct $\underline{\boldsymbol{\sigma}}_h$ in the space $\underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_h)$, see (7.5), on the fine simplicial mesh $\mathcal{S}_h$. For a given $D \in \mathcal{D}_h$ and $\boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h)$ given by (7.37), let

$$\underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^0(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_D);\, \underline{\boldsymbol{v}}_h\mathbf{n}_F = \boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h) \quad \forall F \in \partial\mathcal{S}_h^{\mathrm{int}},\, F \subset \partial D\}.$$

We define $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_h)$ by solving on each $D \in \mathcal{D}_h$ the following minimization problem:

$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^0(\mathcal{S}_D),\, \nabla\cdot\underline{\boldsymbol{v}}_h = -\boldsymbol{\Pi}_0\mathbf{f}} \|\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{v}}_h\|_D. \tag{7.40}$$

Note that as we only have to set the normal fluxes over the side F associated with the given $D \in \mathcal{D}_h$, the linear system (7.40), contrarily to (7.19), is trivial, with a diagonal $d \times d$ matrix; thus a direct flux formula follows from (7.40). We have the following result.

Lemma 7.8 (Reconstructed flux in the nonconforming Crouzeix–Raviart method) *Let* $\mathbf{f}$ *be piecewise constant on* $\mathcal{T}_h$ *and let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be given by* (7.36a)–(7.36b). *Let* $\underline{\boldsymbol{\sigma}}_h$ *be defined by* (7.40). *Then* (4.3) *holds. More precisely,*

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = -\mathbf{f}|_T \qquad \forall T \in \mathcal{S}_h. \tag{7.41}$$

Finally, the next result follows along the lines of the proof of Lemma 7.5.

Lemma 7.9 (Approximation property of the nonconforming Crouzeix–Raviart method) *Let the assumptions of Lemma* 7.8 *be verified. Then the approximation property* (6.2) *holds.*

7.4 Finite volume and related locally conservative methods

This section is devoted to the application of the estimates of Theorems 5.1 and 6.1 to finite volume methods, cf., e.g., [26], and, in a larger sense, to general locally conservative methods such as the mimetic finite difference one, cf., e.g., [11].

A general locally conservative method for the problem (2.3a)–(2.3b) enables to find the side normal fluxes $\boldsymbol{\Upsilon}_F$, constant d-dimensional vectors for each side $F \in \partial\mathcal{T}_h$, such that

$$\sum_{F\in\mathcal{F}_T} \boldsymbol{\Upsilon}_F^i(\mathbf{n}_T{\cdot}\mathbf{n}_F) + (\mathbf{f},\mathbf{e}_i)_T = 0, \qquad i=1,\ldots,d, \quad \forall T\in\mathcal{T}_h. \tag{7.42}$$

Usually, velocities $\mathbf{u}_h \in [\mathbb{P}_0(\mathcal{T}_h)]^d$ and pressures $p_h \in \mathbb{P}_0(\mathcal{T}_h)$ are also obtained from the given numerical scheme.

Suppose first that $\mathcal{T}_h$ is simplicial and matching. In order to obtain a posteriori error estimates in this case, we define a flux (stress) $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{T}_h)$, see (7.5), prescribing its degrees of freedom by

$$\underline{\boldsymbol{\sigma}}_h\mathbf{n}_F|_F = \frac{\boldsymbol{\Upsilon}_F}{|F|} \qquad \forall F\in\mathcal{F}_T, \quad \forall T\in\mathcal{T}_h. \tag{7.43}$$

Then (4.3) immediately follows from (7.42), (7.43), and the Green theorem. Consequently, Theorem 5.1 could directly be applied to obtain an a posteriori error estimate for $|||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||$. As, however, $\mathbf{u}_h \in [\mathbb{P}_0(\mathcal{T}_h)]^d$, $\nabla\mathbf{u}_h$ is a zero tensor and such an estimate would be of very little practical value. We thus, following [50,52], introduce a postprocessed velocity $\tilde{\mathbf{u}}_h \in [\mathbb{P}_2(\mathcal{T}_h)]^d$ satisfying

$$\nabla\tilde{\mathbf{u}}_h|_T - p_h\underline{\boldsymbol{I}}|_T = \underline{\boldsymbol{\sigma}}_h|_T \qquad \forall T\in\mathcal{T}_h, \tag{7.44a}$$

$$\frac{(\tilde{\mathbf{u}}_h,\mathbf{e}_i)_T}{|T|} = \mathbf{u}_h^i|_T, \qquad i=1,\ldots,d, \quad \forall T\in\mathcal{T}_h. \tag{7.44b}$$

Note that such a postprocessing is local on each mesh element T and is cheap, as we merely prescribe the degrees of freedom of $\tilde{\mathbf{u}}_h$. The advantage of this postprocessing is twofold: firstly, $\nabla\tilde{\mathbf{u}}_h$ is no more a zero tensor and it gives a good sense to estimate $|||(\mathbf{u}-\tilde{\mathbf{u}}_h, p-p_h)|||$; secondly, by (7.44a), $\|\nabla\tilde{\mathbf{u}}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T = 0$. Thus, (6.2) (with $\mathbf{u}_h$ replaced by $\tilde{\mathbf{u}}_h$) is trivially satisfied. This is perfectly in agreement with the “flux-conforming” nature of locally conservative methods.

Meshes consisting of general polygons (polyhedrons), possibly nonconvex and not star-shaped, and nonmatching meshes can be taken into account following [52, Section 5]: one introduces a simplicial submesh $\mathcal{S}_T$ of each $T\in\mathcal{T}_h$ such that $\mathcal{S}_T$ form a conforming simplicial mesh $\mathcal{S}_h$ of Ω. One then uses the validity of the balance equation (7.42) on each $T\in\mathcal{T}_h$ in order to solve on the mesh $\mathcal{S}_T$ of each $T\in\mathcal{T}_h$ a local Stokes problem, yielding a side normal flux $\boldsymbol{\Upsilon}_F$ for each side $F\in\partial\mathcal{S}_h$ and $\mathbf{u}_h\in[\mathbb{P}_0(\mathcal{S}_h)]^d$ and $p_h\in\mathbb{P}_0(\mathcal{S}_h)$. Then the approach of the previous paragraph can be applied directly.

Remark 7.4 (Estimates for the fluxes $\underline{\boldsymbol{\sigma}}_h$) Estimates on the error directly in the fluxes $\underline{\boldsymbol{\sigma}}_h$ can be established along the lines of the analysis in [54].

7.5 Mixed finite element methods

Here, we derive locally efficient a posteriori error estimates for mixed finite element methods using Corollary 5.2 and Theorem 6.1. We suppose that $\mathcal{T}_h$ is simplicial and matching.

The mixed finite element method for problem (2.3a)–(2.3b) reads: find $(\underline{\boldsymbol{\sigma}}_h,\mathbf{u}_h,p_h)\in\underline{\boldsymbol{\Sigma}}_h\times\mathbf{V}_h\times Q_h$, the approximation to the stress tensor $\underline{\boldsymbol{\sigma}}$, the velocity $\mathbf{u}$, and the pressure p, respectively, such that

$$(\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\tau}}_h) + (\mathbf{u}_h,\nabla{\cdot}\underline{\boldsymbol{\tau}}_h) = 0 \qquad \forall\underline{\boldsymbol{\tau}}_h\in\underline{\boldsymbol{\Sigma}}_h, \tag{7.45a}$$

$$-(\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h,\mathbf{v}_h) + (\nabla p_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h) \qquad \forall\mathbf{v}_h\in\mathbf{V}_h, \tag{7.45b}$$

$$(\mathbf{u}_h,\nabla q_h) = 0 \qquad \forall q_h\in Q_h. \tag{7.45c}$$

Here, $\underline{\boldsymbol{\Sigma}}_h := \underline{\boldsymbol{\Sigma}}^k(\mathcal{T}_h)$ (see (7.5)), $\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d$, and $Q_h := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap C(\Omega) \cap Q$, $k \geq 0$. Brezzi–Douglas–Marini/Brezzi–Douglas–Durán–Fortin finite element spaces can also be considered instead of the Raviart–Thomas–Nédélec ones, as in [45].

In order to obtain an upper bound on the error $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||$, we could now directly apply Corollary 5.2. Indeed, $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}_h$, so that $\underline{\boldsymbol{\sigma}}_h$ belongs to $\underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$ by definition, and (5.10) follows from (7.45b). As, however, explained in Section 7.4, such a direct application is not too wise. Thus, following [6,5,50,54], we once again introduce a cheap elementwise postprocessing of the velocity $\mathbf{u}_h$. Let $T \in \mathcal{T}_h$ and let $\underline{\boldsymbol{\Sigma}}_h(T)$ denote the restriction of $\underline{\boldsymbol{\Sigma}}_h$ onto T and similarly for $\mathbf{V}_h(T)$. We look for $\tilde{\mathbf{u}}_h|_T \in \mathbf{M}_h(T)$ such that

$$(\nabla \tilde{\mathbf{u}}_h - \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_T = 0 \qquad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h(T), \tag{7.46a}$$

$$(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h(T). \tag{7.46b}$$

Equivalently, the above definition can be expressed as

$$\underline{\boldsymbol{\Pi}}_{\underline{\boldsymbol{\Sigma}}_h(T)}(\nabla \tilde{\mathbf{u}}_h|_T) = \underline{\boldsymbol{\sigma}}_h|_T, \tag{7.47a}$$

$$\boldsymbol{\Pi}_{\mathbf{V}_h(T)}(\tilde{\mathbf{u}}_h|_T) = \mathbf{u}_h|_T, \tag{7.47b}$$

where $\underline{\boldsymbol{\Pi}}_{\underline{\boldsymbol{\Sigma}}_h(T)}$ is the $\underline{\boldsymbol{L}}^2$-orthogonal projection onto $\underline{\boldsymbol{\Sigma}}_h(T)$ and $\boldsymbol{\Pi}_{\mathbf{V}_h(T)}$ is the $\mathbf{L}^2$-orthogonal projection onto $\mathbf{V}_h(T)$. The spaces $\mathbf{M}_h(T)$ are vector variants of those of [6,5]. They are typically $[\mathbb{P}_{k+1}(\mathcal{T}_h)]^d$ spaces enriched by bubbles. Moreover, in the lowest-order case ($k = 0$), as in (7.44a)–(7.44b) (cf. also (7.22a)–(7.22b)), following [50], one can easily build $\tilde{\mathbf{u}}_h$ such that

$$\nabla \tilde{\mathbf{u}}_h|_T = \underline{\boldsymbol{\sigma}}_h|_T \qquad \forall T \in \mathcal{T}_h, \tag{7.48a}$$

$$\frac{(\tilde{\mathbf{u}}_h, \mathbf{e}_i)_T}{|T|} = \mathbf{u}_h^i|_T, \qquad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h. \tag{7.48b}$$

We then apply Corollary 5.2 in order to estimate $|||(\mathbf{u} - \tilde{\mathbf{u}}_h, p - p_h)|||$. In the lowest-order case ($k = 0$) and constructing $\tilde{\mathbf{u}}_h$ by (7.48a)–(7.48b), $\|\nabla \tilde{\mathbf{u}}_h - \underline{\boldsymbol{\sigma}}_h\|_T = 0$. Hence in this case, (6.3) is trivially satisfied, once again in agreement with the flux-conforming nature of mixed finite elements. For $k \geq 1$, this property does not hold exactly anymore. By (7.47a), however, $\|\nabla \tilde{\mathbf{u}}_h - \underline{\boldsymbol{\sigma}}_h\|_T$ is expected to be small and act as a numerical quadrature. Finally, we note that Remark 7.4 applies here as well. Proceeding as in [54], rigorous both-sided estimates, also including the estimates on the error directly in the fluxes $\underline{\boldsymbol{\sigma}}_h$, can be obtained.

8 Numerical experiments

In this section, we illustrate the theory using discontinuous, conforming, and nonconforming methods. As a discontinuous method, we consider first- and second-order symmetric discontinuous Galerkin (DG) method of Section 7.1, i.e., (7.2) with $\theta = 1$ and $k = 1, 2$. The conforming example will be computed using the $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ method of Section 7.2, and the nonconforming example using the Crouzeix–Raviart method of Section 7.3.

The a posteriori error estimates for these methods are obtained by recovering the equilibrated flux $\underline{\boldsymbol{\sigma}}_h$ and applying Theorem 5.1 or Corollary 5.2, depending on the method. For the discontinuous Galerkin method of order k, we recover the flux from the space $\underline{\boldsymbol{\Sigma}}^k(\mathcal{T}_h)$. The error estimator is obtained by applying Theorem 5.1. For such flux and sufficiently regular $\mathbf{f}$, Lemma 7.1 guarantees superconvergence for the residual error estimators $\eta_{\mathrm{R},T}$. As we will see, this is not true if $\mathbf{f}$ is not sufficiently regular. For the $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ and Crouzeix–Raviart methods, the flux is recovered by solving local minimization problems (7.19) and (7.40), respectively. In order for Lemmas 7.4 and 7.8 hold, we, as usual, implement these methods with $\mathbf{f}$ replaced by $\boldsymbol{\Pi}_0\mathbf{f}$, where $\boldsymbol{\Pi}_0$ is the $\mathbf{L}^2$-orthogonal projection onto $[\mathbb{P}_0(\mathcal{T}_h)]^d$. We then include the data oscillation in the residual error estimators $\eta_{\mathrm{R},T}$ (5.8) and (5.3) as $C_{\mathrm{P},T} h_T \|\mathbf{f} - \boldsymbol{\Pi}_0\mathbf{f}\|_T$, which are once again superconvergent for smooth $\mathbf{f}$. The error estimate for the $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ method is obtained by applying Corollary 5.2 and for the Crouzeix–Raviart method by applying Theorem 5.1.

Throughout this section, we will consider domain $\Omega = (0,1)^2$. To evaluate the energy (semi-)norm (3.5) and the divergence error estimator $\eta_{\mathrm{D},T}$ of (5.2) the inf–sup constant β has to be estimated. Although the inf–sup constant can be estimated analytically for rectangular domains, we have computed β with the procedure from [24]. Based on these computations, the value $\beta = 0.44$ is used.

The load function $\mathbf{f}$ is chosen to correspond to the solution

$$\mathbf{u} = \nabla \times (x-1)^2 x^{1+\alpha}(y-1)^2 y^2 \mathbf{e}_3, \quad p = x + y - 1. \tag{8.1}$$

For $\alpha > 0$, the velocity field $\mathbf{u}$ has a zero divergence, $\nabla \cdot \mathbf{u} = 0$, and satisfies the zero Dirichlet boundary condition, $\mathbf{u} = 0$ on $\partial\Omega$. The regularity of $\mathbf{u}$ is $[H^{\frac{1}{2}+\alpha}(\Omega)]^d$ for $\alpha \notin \mathbb{N}$ and $[C^\infty(\Omega)]^d$ for $\alpha \in \mathbb{N}$.

For all methods, we will first consider problem with a smooth solution. For this purpose, the parameter α is chosen as $\alpha = 1$. To compare the error estimator with the exact error in uniform refinement, we have solved the problem at hand with each of the mentioned methods on a set of uniformly refined meshes. The error and estimates behavior for the different schemes are visualized in Figure 8.1. The corresponding effectivity indices, given as the ratios of the estimate over the error, are presented in Figure 8.2. The different estimators, namely

$$\left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2}, \tag{8.2}$$

with $\eta_T = \eta_{\mathrm{NC},T}, \eta_{\mathrm{DF},T}, \eta_{\mathrm{R},T}$, and $\eta_{\mathrm{D},T}$, are plotted in Figure 8.3. For each method, the predicted superconvergence for the $\eta_{\mathrm{R},T}$ part is observed.

The error distributions from refinement step 5 for discontinuous Galerkin methods, the $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ method, and the Crouzeix–Raviart method are given respectively in Figures 8.4, 8.5, 8.6, and 8.7. As the error bounds given in Theorem 5.1 or Corollary 5.2 are not in an elementwise form, we have estimated the upper bound as

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||^2 \le 2 \sum_{T \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},T}^2 + 12(\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2 \right\},$$

using also the bound (3.7) on the constant C_{S}; we have used the term $\eta_T := \{2(\eta_{\mathrm{NC},T}^2 + 12(\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2)\}^{1/2}$ as elementwise error estimator. Note that the estimated and exact distributions match very well.

In the second example, we set $\alpha = 0.75$, so that the velocity $\mathbf{u}$ is in $[H^{1.25}(\Omega)]^d$ and there is a boundary singularity on the edge $x = 0$. In this example, all computations were performed using the first-order DG method. The problem was solved either on uniformly refined meshes or using a simple adaptive procedure. In the adaptive routine, we refine ten percents of elements in each step. The elements are chosen such that they have the largest element estimators η_T.

The error behavior in the adaptive and uniform refinement procedures is visualized in Figure 8.8 and the corresponding effectivity index in Figure 8.9. The adaptive refinement procedure has considerably faster, optimal with respect to the number of mesh elements, convergence rate compared to the uniform mesh refinement, where the convergence rate is in agreement with the $[H^{1.25}(\Omega)]^d$ regularity of the weak solution ($h^{0.25}$). As in the smooth test case, the error estimate overestimates the error, but decreases with the same speed as the exact error. Figure 8.10 shows the estimated and exact error distributions in this case. They once again match very well; in particular the boundary singularity is well detected. The superconvergence of $\eta_{\mathrm{R},T}$ does not appear anymore as $\mathbf{f}$ is not sufficiently regular. For the sake of completeness, initial, third, and fifth adaptive mesh are visualized in Figure 8.11. One can observe the expected refinement towards the boundary singularity.

A Appendix: the inf–sup condition on $\mathbf{V} \times Q$

We prove in this appendix Lemma 3.1, giving the Babuška–Brezzi splitting for the Stokes problem. We proceed in five steps.

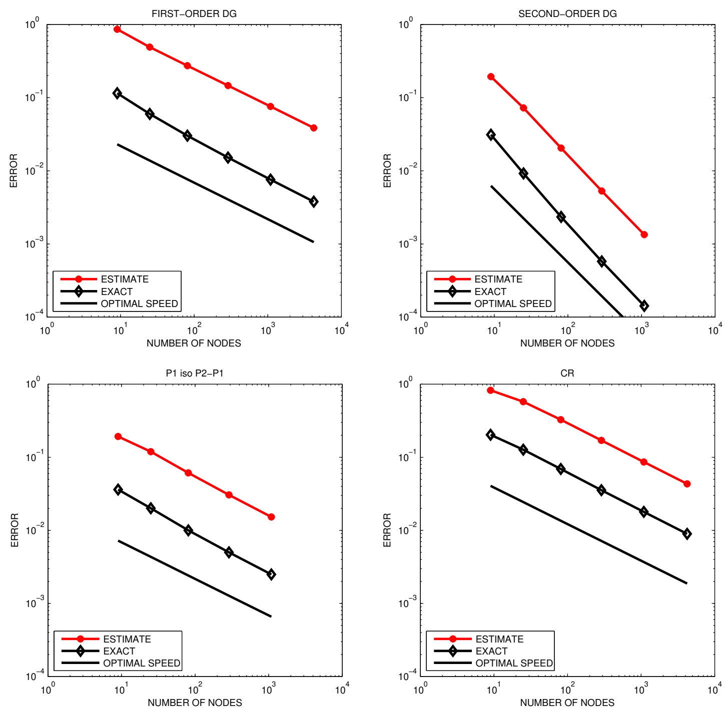

Fig. 8.1 Estimated and exact errors for the first-order DG, second-order DG, $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$, and Crouzeix–Raviart methods for the smooth test case

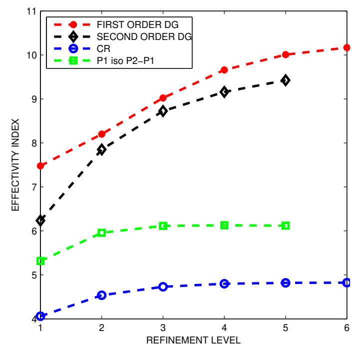

Fig. 8.2 Effectivity indices for the first-order DG, second-order DG, $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$, and Crouzeix–Raviart methods for the smooth test case

1) It is immediate to see from the definition (3.3) of $\mathcal{B}$ and (2.2a) of a that for an arbitrary $(\mathbf{v}, q) \in \mathbf{V} \times Q$,

$$\mathcal{B}((\mathbf{v}, q), (\mathbf{v}, -q)) = \|\nabla \mathbf{v}\|^2.$$

2) The inf–sup condition (2.4), using a scaling argument, ensures that for a given $q \in Q$, there exists $\mathbf{w} \in \mathbf{V}$ such that

$$(q, \nabla \cdot \mathbf{w}) \geq \beta \|q\|^2 \text{ and } \|\nabla \mathbf{w}\| = \|q\|.$$

Recall also the arithmetic–geometric–mean inequality:

$$-|ab| \geq -\frac{\varepsilon}{2} a^2 - \frac{1}{2\varepsilon} b^2, \qquad \forall a, b \in \mathrm{R},\ \forall \varepsilon > 0.$$

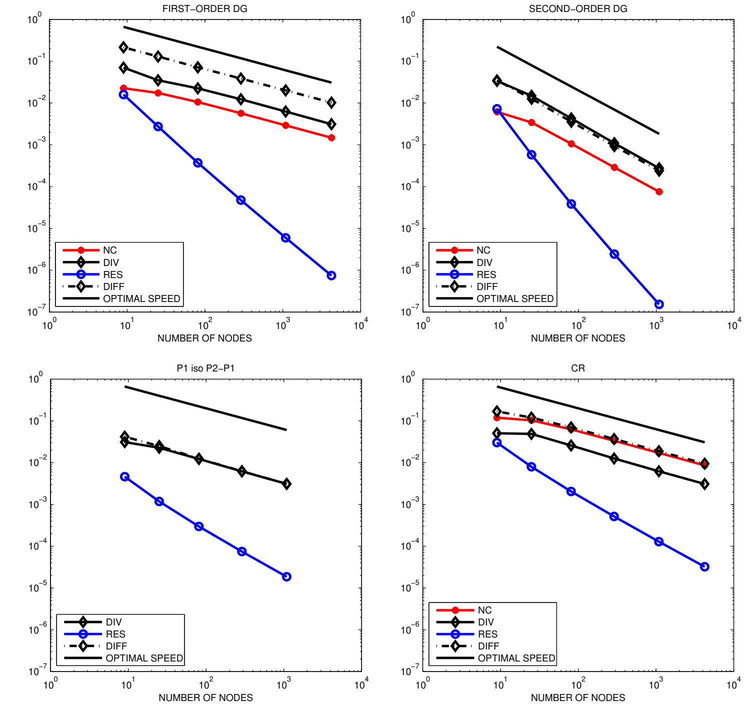

Fig. 8.3 Different estimators for the first-order DG, second-order DG, $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$, and Crouzeix–Raviart methods for the smooth test case

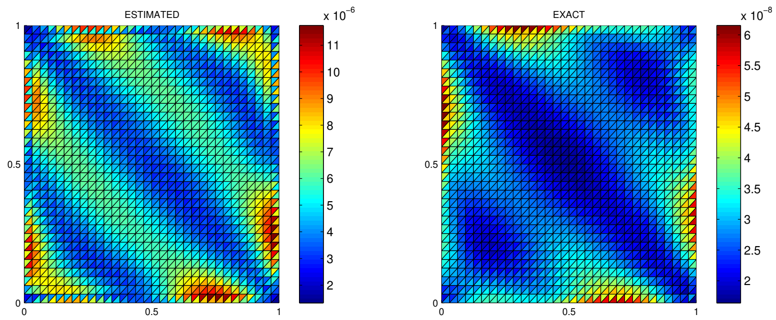

Fig. 8.4 Estimated (left) and exact (right) error distributions for the first-order DG method in the smooth test case

Using these ingredients and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\mathcal{B}((\mathbf{v},q),(-\mathbf{w},0)) &= -(\nabla\mathbf{v},\nabla\mathbf{w}) + (q,\nabla\cdot\mathbf{w}) \geq -\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\| + \beta\|q\|^2 \\ &\geq -\frac{1}{2\beta\varepsilon}\|\nabla\mathbf{v}\|^2 + \beta\Big(1-\frac{\varepsilon}{2}\|q\|^2\Big).\end{aligned}$$

3) For $\delta > 0$ arbitrary, we have, using the results of steps 1) and 2),

$$\begin{aligned}\mathcal{B}((\mathbf{v},q),(\mathbf{v}-\delta\mathbf{w},-q)) &= \mathcal{B}((\mathbf{v},q),(\mathbf{v},-q)) + \delta\mathcal{B}((\mathbf{v},q),(-\mathbf{w},0)) \\ &\geq \|\nabla\mathbf{v}\|^2 - \frac{\delta}{2\beta\varepsilon}\|\nabla\mathbf{v}\|^2 + \beta\delta\Big(1-\frac{\varepsilon}{2}\|q\|^2\Big).\end{aligned}$$

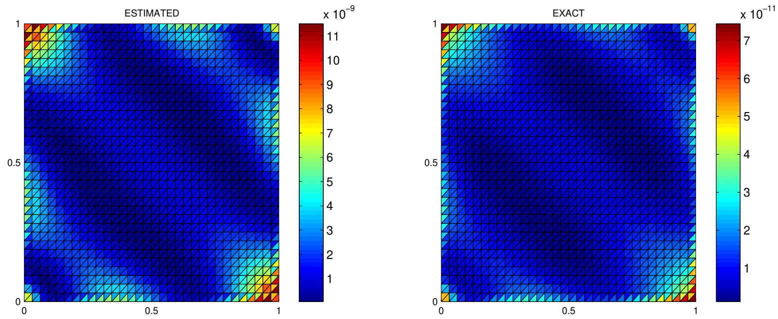

Fig. 8.5 Estimated (left) and exact (right) error distributions for the second-order DG method in the smooth test case

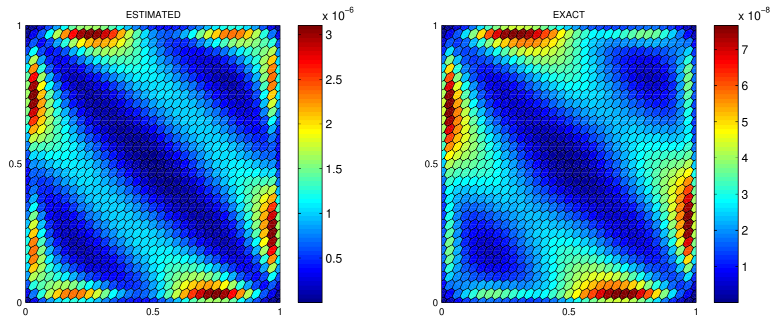

Fig. 8.6 Estimated (left) and exact (right) error distributions for the $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ method in the smooth test case

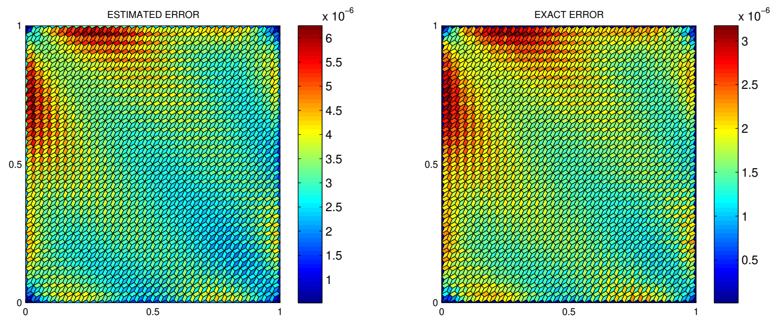

Fig. 8.7 Estimated (left) and exact (right) error distributions for the nonconforming Crouzeix–Raviart method in the smooth test case

Choosing $\delta = \beta$ and $\varepsilon = 1$, we arrive at

$$\mathcal{B}((\mathbf{v}, q), (\mathbf{v} - \beta \mathbf{w}, -q)) \geq \frac{1}{2}\|\nabla \mathbf{v}\|^2 + \frac{1}{2}\beta^2\|q\|^2 = \frac{1}{2}|||(\mathbf{v}, q)|||^2.$$

4) For $\mathbf{z} = \mathbf{v} - \beta \mathbf{w}$ and $r = -q$, we thus have

$$\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r)) \geq \frac{1}{2}|||(\mathbf{v}, q)|||^2.$$

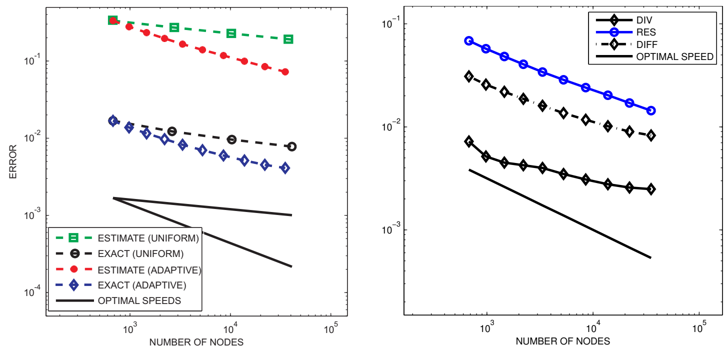

Fig. 8.8 Estimated and exact errors in uniform/adaptive refinement (left) and components of the error estimator in adaptive refinement (right) for the first-order DG method in the singular test case

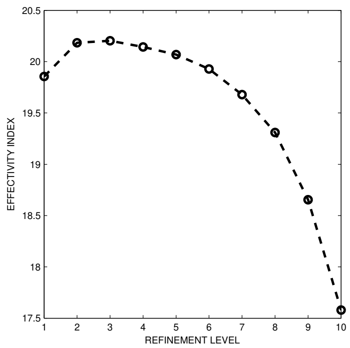

Fig. 8.9 Effectivity indices for the first-order DG method in the singular test case

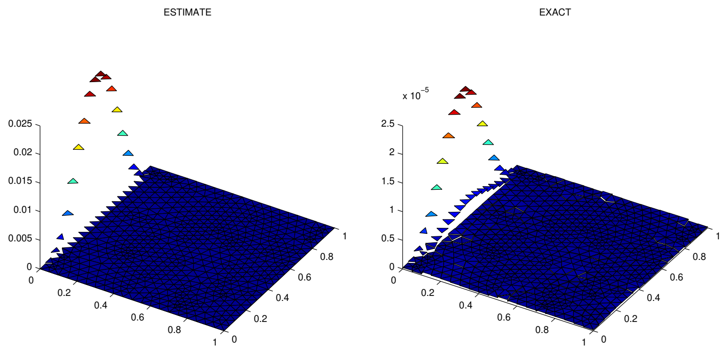

Fig. 8.10 Estimated (left) and exact (right) error distributions for the first-order DG method in the singular test case

Next,

$$
\begin{aligned}
|||(\mathbf{z}, r)|||^2 &= \|\nabla(\mathbf{v} - \beta \mathbf{w})\|^2 + \beta^2 \|q\|^2 \\
&\le 2\|\nabla \mathbf{v}\|^2 + 2\beta^2 \|\nabla \mathbf{w}\|^2 + \beta^2 \|q\|^2 \\
&\le 3(\|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2) \\
&= 3|||(\mathbf{v}, q)|||^2.
\end{aligned}
$$

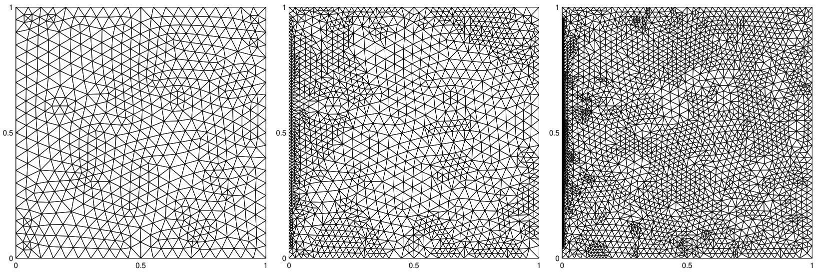

Fig. 8.11 Initial (left), third (middle), and fifth (right) mesh for the adaptive refinement for the first-order DG method in the singular test case

5) Combining the results of the previous steps, we see that for an arbitrary $(\mathbf{v}, q) \in \mathbf{V} \times Q$, there exists $(\mathbf{z}, r) \in \mathbf{V} \times Q$ such that

$$\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r)) \geq \frac{1}{2} \frac{1}{\sqrt{3}} |||(\mathbf{v}, q)|||\ |||(\mathbf{z}, r)|||,$$

which is precisely the statement of the lemma.

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